Negative Income Shocks and Asset Pricing

Steven Kou† and Seyoung Park‡

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Keywords: Income Shock, Incomplete Market, General Equilibrium, Asset Pricing

JEL Codes: D15, D58, G11, G12

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1 Introduction

1.1 Background

Potentially catastrophic loss of income is now an omnipresent risk. The impact of the COVID-19 pandemic has exposed working members of society to large and negative income shocks, driving increased concerns about employment loss risk, and more generally about discontinuities in income flow.[1] Insecurity and volatility levels around earnings are expected to increase due to this pandemic and affect many more individuals over their life cycle.[2] Equally, half of jobs in the world are susceptible to becoming automated in the future (Frey and Osborne, 2017). This is another risk to labor markets which is expected to significantly disrupt a large cross-section of our society. Considering such potentially catastrophic income losses, a very clear need has arisen to research risk management strategies to alleviate potential discontinuities in the income stream.[3]

To cope with disastrous income shocks arising from rare events by understanding optimal risk management practices, our focus is to generalize Friedman's (1957) permanent income hypothesis (PIH) with a large, negative income shock (LNIS) and then to evaluate whether the generalized framework can explain how people would respond to the LNIS to increase their resilience to it.[4] Having generalized the PIH with the LNIS, we then examine

[1] The current COVID-19 pandemic has increased the possibility of losing substantial part of individuals' income resulting from potential bankruptcies of firms which they work for.

[2] In the US, 44% of households could not cover an emergency expense of only $400, so they will struggle if they experience an unexpected hardship (Federal Reserve report, 2017). For more details, refer to "Report on the Economic Well-Being of U.S. Households in 2016" published by Board of Governors of the Federal Reserve System on May 2017. In the European Union, approximately 218 million people are experiencing earnings insecurity and volatility and struggling to ensure that future consumption needs can be met (European Commission statistics, 2017). For more details, refer to EU Statistics on Income and Living Conditions (EU-SILC 2017).

[3] Otherwise, the failure to provide appropriate income protection for businesses against the long term disruptive economic impact of the pandemic would result in business failures and costly bankruptcy (Milne, 2020).

[4] The research findings will provide insights and understanding to individuals for determining how to

the ability of an asset pricing model with the LNIS to explain the equity premium puzzle and the risk-free rate puzzle especially through the key economic mechanism of optimal consumption/savings. To obtain analytically tractable general equilibrium quantities, we consider a pure exchange economy in the type of Lucas (1978). The generalization is that a representative agent's aggregate output is exposed to the LNIS. To investigate the effects of the LNIS in our equilibrium setting, we have derived Euler equation and general equilibrium risk-free rate and equity premium in closed-form.[5]

1.2 Contribution

In this paper, we provide a simple explanation via Friedman's PIH with the LNIS to fit both low risk-free rate and high equity premium especially by using a small number of parameters. The key economic mechanism understanding the PIH channel is that the Arrow-Debrue price (or the shadow price) further increases with the LNIS (Theorem 4.1 in Section 4). The increased price could influence an increase in the equilibrium consumption price, implying that the amount of present consumption the agent would be willing to give up now to receive one more unit of future consumption becomes larger than without the LNIS. To continue being able to afford with the more expensive consumption price what the agent can currently afford, she would then significantly increase her optimal savings with the LNIS when wealth is large as well (Figure 3 in Section 5). The demand for optimal savings is sufficiently strong making the agent save at a high rate[6] and hence, lowering the risk-free rate significantly, which is particularly relevant to today's low-interest-rate environment. Such a savings decision discourages equity investment, so the equity premium increases.

continue being able to afford what they can currently afford, i.e., how to attain a smooth profile of future consumption.

[5] In a general equilibrium sense, we also generalize in a rational expectations equilibrium framework the Ramsey rule (Ramsey, 1928) showing that interest rate equals subjective discount rate plus the product of consumption elasticity of marginal utility and consumption growth rate.

[6] The LNIS faced by the agent causes her likely to reduce consumption to secure extra reserves in preparation for financing future consumption needs by using her savings.

Contrary to the standard asset pricing model without the LNIS, in our framework high values of risk aversion no longer counterfactually generate high risk-free rates. Rather, an increase in risk aversion results in a decrease in the risk-free rate (Figure 1 in Section 5), thereby helping to disentangle the risk-free rate and equity premium and thus, avoiding the risk-free rate puzzle (Weil, 1990).

The equity premium and the risk-free rate are determined in equilibrium and matched up with the century-long sample (1891-1994) by Campbell (1999) and the long historical sample (1871-2011) by the website of Robert Shiller (http://www.econ.yale.edu/∼shiller/data/chap26.xls). The risk-free rates and the equity premia generated by our asset pricing model with the LNIS are exactly the same with those observed from the century-long sample and the long historical sample, requiring empirically plausible parameter values (Table 1 in Section 5).

Technically, our model improves methods proposed by Cox and Huang (1989) and Karatzas *et al.* (1991) (who do not consider income) to solve optimal consumption/investment and asset pricing problems with large income shocks. The improvement that we have in Section 3 and Section 4 essentially uses ideas suggested by Bensoussan *et al.* (2016) to solve incomplete-market problems with income risk. We then extend with the solutions the asset pricing framework of Lucas (1978) to allow for income shocks.

1.3 Literature Review and Outline

Permanent Income Hypothesis. The permanent income hypothesis (PIH) of Friedman (1957) suggests that consumption is proportional to the sum of financial wealth and human capital, which is the discounted expected value of future income at the risk-free interest rate. So, the changes in marginal consumption with financial wealth or human capital, i.e., when future wealth or income shocks are possible, individuals' optimal decision is to save more to attain consumption smoothing (a smooth profile of future consumption). In line with Friedman's PIH intuition, our generalized PIH with the LNIS suggests that responding to the LNIS would require substantial precautionary savings for consumption smoothing.

Individual Savings. According to the PIH predictions of Bewley (1977) and Campbell (1987), an income shock is less likely to affect the optimal savings of people who are at the higher end of wealth. This is because consumption of the wealthy can be financed mainly by wealth, without resorting to income. Intuitively, the ability to self-insure against the income shock improves when wealth is large, so the optimal savings decrease as wealth increases and then turns negative if wealth is very large relative to income. Contrary to Bewley (1977) and Campbell (1987), a savings motive for precautionary reasons in the event of the LNIS has a first-order effect on the optimal savings decision of wealthy people. Empirical and anecdotal evidence shows that positive and even high savings rates are very common amongst wealthy people. Indeed, wealthy people are inclined to exhibit high savings rates as follows: a positive relation between savings rates and income (Dynan *et al.*, 2004), entrepreneurship purposes for entering and expanding business (Buera, 2009), out-of-pocket medical expenses patterns (De Nardi *et al.*, 2010), the mix of bequests and human capital, entrepreneurship, and medical-expense risk (De Nardi and Fella, 2017). Our generalized PIH with the LNIS can partially support high savings rates of wealthy people.

Optimal Consumption and Investment. Our work sits squarely within the optimal consumption and investment framework. Starting from the seminal work of Merton (1969, 1971), many studies have incorporated nontradable income in the framework.[7] Importantly, the (undiversifiable) labor income risk has become a standard element in studies of optimal strategies. When modeling and interpreting labor income shocks, standard literature has assumed log-normality with a Brownian motion.[8] However, large and negative income

[7]For instance, refer to Farhi and Panageas (2007) and Jang *et al.* (2013).

[8]For example, Merton (1971), Bodie *et al.* (1992), Duffie *et al.* (1997), Koo (1998), Cocco *et al.* (2005), Gomes and Michaelides (2005), Polkovnichenko (2007), Benzoni *et al.* (2007), Wachter and Yogo (2010), Dybvig and Liu (2010), Munk and Sørensen (2010), Lynch and Tan (2011a, 2011b), Calvet and Sodini (2014), Ahn *et al.* (2019), and Jang *et al.* (2019).

shocks affect individual life-cycle strategies significantly, and cause substantial deviations from log-normality (Guvenen *et al.*, 2015). Furthermore, models that use the Brownian motion cannot appropriately account for the effects of low-probability, high-impact events such as forced unemployment and job displacement.[9] To reflect the empirical reality, we consider the LNIS in the optimal consumption and investment framework.

Incomplete Market. It has long been known that market completeness under no arbitrage implies the existence of a unique state price density and the resulting unique risk-neutral measure, under which the expected return on any asset becomes the risk-free rate (Ross, 1978). However, when markets are incomplete, i.e., when risk may be undiversifiable, the assumption of risk-neutrality with the unique state price density cannot be justified. Rather, the number of state price densities is infinite, so the set of equivalent martingale measures is also infinite. To price the expected return on an asset under the LNIS, this multitude of state price densities must be pruned to one.

Although a theoretical framework for the martingale pricing in incomplete markets exists (Karatzas *et al.*, 1991), it cannot easily be used to characterize the set of state price densities explicitly.[10] The state price density has been derived explicitly by Kou (2002) and Liu et al. (2003). However, these models have overlooked the labor income risk, which is a major dimension of market incompleteness.[11]

[9] Low et al. (2010) show that large earnings losses are observed at job displacement.

[10] In order to address the challenges of market incompleteness, instead of the martingale pricing approach, alternative dynamic programming approach can be used for the pricing in incomplete markets (Duffie *et al.*, 1997; Liu et al., 2005). However, in this case, it involves highly non-linear Hamilton-Jacobi-Bellman (HJB) equations, which are almost impossible to solve analytically. Therefore, use of dynamic programming approach requires use of complex numerical schemes to solve incomplete market problems. With no consideration of labor income and its risk, one can adopt the approaches of Garlappi and Skoulakis (2010), Jin and Zhang (2012), and Jin et al. (2017) for such a numerical approach to solving the consumption and investment problem in incomplete markets.

[11] Based on the idea of market completion by Karatzas *et al.* (1991), Liu *et al.* (2003) have established a dynamically completed market with derivatives. Similar to Liu *et al.* (2003), Branger et al. (2017) add derivatives to complete the market in which variances and covariances of return are stochastic. It would

In our work, the LNIS is driven by an exogenous shock that is assumed to occur with a Poisson probability distribution, and hence cannot be fully diversified away. In such a setting with the random arrival of the income shock, the classical martingale pricing approach (Cox and Huang, 1989) that uses the risk-neutral measure is no longer available. We believe the main difficulty lies in that allowing for market incompleteness usually gives rise to considerable challenges in deriving the closed-form (or analytically tractable) unique state price density for pricing purposes. Ours is a first attempt to develop an analytically tractable martingale pricing approach in an incomplete market with the LNIS. We start from the idea of fictitious completion (Karatzas *et al.*, 1991) to establish a risk-neutral intensity of the LNIS; the uniquely determined risk-neutral intensity should be used when deriving the dynamic budget constraint.[12]

Equity Premium Puzzle. Mehra and Prescott (1985) initially raise the so-called equity premium puzzle, which has been dubbed by Campbell (1999) as the following question: why is the average real stock return so high in relation to the average short-term real interest rate? There is an extensive literature in an attempt to resolve the equity premium puzzle within a rational expectations equilibrium framework especially in a complete market setting: Basak (1995), Heaton and Lucas (1996), Basak and Cuoco (1998), Basak and Shapiro (2001), Liu et al. (2003), Maenhout (2004), Gârleanu and Panageas (2015), Kimball *et al.* (2018), Gomez (2019), Gârleanu and Panageas (2019).

Amongst various economic channels that matter for asset prices, unhedgeable/uninsurable idiosyncratic income shocks have been considered in standard asset pricing models to match the equity premium: Grossman and Shiller (1982), Lucas (1994), Wang (2003), Gomes and Michaelides (2008), Guvenen (2009), Kreuger and Lustig (2010), Christensen *et al.* (2012), Dumas and Lyasoff (2012), Schmidt (2016), Constantinides and Ghosh (2017). Due to the

be tricky to apply their models to the incomplete market setting, without resorting to the idealistically completed market.

[12]More recently, Ahn et al. (2019) provide a numerical scheme to characterize the unique state price density.

limitations of conventional income shock modeling in a diffusive-type shock (Brownian-risk) setting, the existing models are limited to fully investigate the asset pricing implications of market incompleteness. We present an analytically tractable asset pricing framework with both diffusive income shocks and LNIS in the simplest possible economic setting (without resorting to complex dimensions such as recursive utility and heterogeneity considered by Schmidt (2016) and Constantinides and Ghosh (2017)) and explain both low risk-free rate and high equity premium via a simple Friedman's PIH optimal savings channel.

Risk-Free Rate Puzzle. Weil (1990) first identifies the so-called risk-free rate puzzle. Requiring a very high risk aversion in a response to the equity premium puzzle results in a very high interest rate as well, which has been at odds with the low interest rate we have observed. In an attempt to resolve the risk-free rate puzzle, Constantinides *et al.* (2002) show that borrowing constraints preventing young people from borrowing by capitalizing their human capital can dramatically lower the risk-free rate, thereby partially explaining the risk-free rate puzzle. Bansal and Yaron (2004) establish the long-run risk model in a representative agent economy with recursive utility where the low risk-free rate is generated matching the historical rate.

A few papers have investigated a role of non-hedgeable income shocks in the risk-free rate: Weil (1992), Kreuger and Lustig (2010), Christensen *et al.* (2012). Weil (1992) obtains a sufficient condition in a two-period model under which undiversifiable labor income risk decreases the risk-free rate and increases the equity premium. Kreuger and Lustig (2010) obtains the result in a standard incomplete markets endowment economy just as Lucas (1994) that idiosyncratic labor income risk is known to decrease the risk-free rate. Christensen *et al.* (2012) derive the result in a Brownian-risk setting that unspanned income risk can play a pivotal role in a decrease of the risk-free rate. In a standard infinite horizon, incomplete market, continuous-time setting with the LNIS, we also have the result that the risk-free rate becomes lower with higher risk aversion and larger the LNIS, consistent with the theoretical evidence by Weil (1992), Kreuger and Lustig (2010), Christensen *et al.* (2012). More importantly, we can exactly match the historical risk-free rate with the LNIS-induced precautionary savings generalizing Friedman's PIH, thereby explaining the risk-free rate puzzle.

Related Literature. This paper builds on two strands of the literature. First, it draws on the effects of a permanent discrete-jump income shock on optimal consumption and savings by Wang *et al.* (2016) and Bensoussan *et al.* (2016). Wang *et al.* (2016) study an incomplete-market consumption-savings model with recursive utility and stochastic income modeled by both a Brownian motion and large jump income shocks. Bensoussan *et al.* (2016) study a model of optimal consumption/savings, investment, and retirement with jump-type forced unemployment risk. Our paper incorporates a disastrous income shock in Friedman's PIH, and thoroughly investigates its crucial role in optimal consumption/savings, and, more importantly, in asset prices.

The paper draws on the literature on asset pricing with unhedgeable idiosyncratic income shocks. In contrast to the insignificance of the role of labor income risk in asset prices (Grossman and Shiller, 1982; Krueger and Lustig, 2010), Constantinides and Duffie (1996), Gomes and Michaelides (2008), Guvenen (2009), Christensen *et al.* (2012), Dumas and Lyasoff (2012), Constantinides and Ghosh (2017) have improved their empirical predictions for asset prices by focusing on unhedgeable idiosyncratic income shocks. All of these asset pricing models acknowledge the limitations of conventional income shock modeling in a Brownian-risk setting in matching asset prices. Different from these studies, generalizing Constantinides and Duffie (1996), Schmidt (2016) thoroughly investigates asset pricing implications of idiosyncratic tail (jump) risk in consumption growth and income with recursive preferences, income skewness, heterogeneous agents, and incomplete markets.

Here our goal is different from Schmidt (2016): rather than include all the complex dimensions that are realistic and important in financial and labor markets, this paper isolates and very closely investigates the consumption and savings issues introduced by the LNIS on asset prices. While being motivated by the pivotal role of a large uninsurable component of labor income in asset prices, we further draw on the PIH-based general equilibrium model by Wang (2003) with the LNIS, which is a big step to study the PIH explanations with the LNIS to the equity premium puzzle and the risk-free rate puzzle. More precisely, ours addresses the level of equity premium and risk-free rate by newly finding a substantial precautionary savings motive driven by the LNIS that turns out to do matter for asset prices.

Outline. This paper is organized as follows. In Section 2, we establish the optimal consumption and investment problem with the LNIS. In Section 3, we generalize the PIH and obtain analytically tractable optimal consumption and investment strategies through which the optimal savings required for consumption smoothing are quantified. In Section 4, we provide a general equilibrium analysis with all the equilibrium quantities derived in closed-form. In Section 5, we perform an in-depth quantitative analysis to discuss the impact of the optimal strategies on asset prices and match the equity premium and risk-free rate observed from the data. In Section 6, we conclude the paper.

2 The Model

Model Setup. Uncertainty is driven by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, in which a multi-dimensional Brownian motion and a Poisson process are defined. All stochastic processes are adapted to $\{\mathcal{F}_t\}$, which is the P-augmentation of the filtration generated by the Brownian motion and the Poisson process. All stated stochastic processes are assumed to be well defined, without explicitly stating the regularity conditions.

Financial assets in the market are summarized by one riskless bond and multiple risky stocks. The bond price B and the stock prices S are given by

$$dB(t) = rB(t)dt$$

and

$$dS(t) + D(t) = S(t)\{\mu dt + \sigma^{\top} dZ(t)\}, \tag{1}$$

where r is the risk-free interest rate, $D(t) = (d_1, ..., d_N)$ are dividends for N risky stocks, μ is the constant mean vector, σ is the constant nonsingular standard deviation matrix, and $Z(t)$ is the standard Brownian motion process with dimensionality equal to the number of linearly independent returns on stocks. In the context of finance interpretation, $Z(t)$ represents variations in market/economic condition that stem from a source of market risk (or a market factor) in the economy.

We consider the aggregate output process $I(t)$ modeled by a geometric Brownian motion with a Poisson shock as follows:

$$dI(t) = \mu^I I(t-)dt + (\sigma^I)^\top I(t-)dZ(t) - (1-k)I(t-)dN(t), \quad I(0) = I > 0, \tag{2}$$

where μ^I is the output mean vector and σ^I is the standard deviation vector, $Z(t)$ is the market factor, $k \in (0,1)$ is the recovery parameter, and $N(t)$ is a Poisson shock with intensity $\delta > 0$. The output is exposed to a large, negative income shock (LNIS) represented by the Poisson shock. When the LNIS occurs at time $t-$, the aggregate output plummets immediately to $kI(t-)$ from $I(t-)$.

We assume that the fraction $\xi \in (0,1)$ of aggregate output constitutes aggregate earnings $\xi I(t)$.[13] The remaining fraction $1-\xi$ of aggregate output is paid out as a dividend as:

$$D(t) = (1-\xi)I(t) = I(t) - \xi I(t),$$

which shows that the dividend itself is affected by the LNIS not only in the aggregate output $I(t)$, but also in the aggregate earnings $\xi I(t)$. Such an influence of the LNIS would, thus, affect asset returns in (1) as well.[14]

We consider an infinite-horizon economy with a single consumption good (the numeraire). Each representative agent has wealth $W(t)$ and invests $\pi(t)$ in the stock market, and saves her remaining wealth $W(t) - \pi(t)$ in the bond market. The agent also con-

[13]The aggregate earnings are the total of all earnings in an economy expressing the proceeds from the aggregate (total) output in the economy for producers of that output.

[14]Aggregate earnings risks and asset return risks are closely related (Ball et al., 2009). For instance, there is a positive relation between earnings and asset returns at the firm level (Ball and Brown, 1968).

sumes $c(t)$ and receives $\xi I(t)$. The agent's dynamic wealth (budget) constraint is then:[15] $W(0) = w > -\xi I/\beta_1$,

$$dW(t) = \{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top\sigma^\top dZ(t), \tag{3}$$

where $\mathbf{1}$ is a vector of one's with dimensionality equal to the number of stocks, $\pi(t)$ is the dollar amount vector invested in each risky stock, and the borrowing limit is imposed by the following wealth constraint:[16]

$$W(t) > -\frac{\xi I(t)}{\beta_1}, \quad \text{for all } 0 \leq t < \tau, \tag{4}$$

where

$$\beta_1 = r - \mu^I + (\sigma^I)^\top\theta,\ \theta = (\sigma^\top)^{-1}(\mu - r\mathbf{1}),$$

and τ is the random arrival of the LNIS driven by Poisson shock $N(t)$. Notice that the agent is being exposed to the large, negative wealth shock due to the LNIS in the aggregate earnings $\xi I(t)$ stemming from the random arrival of the Poisson shock $N(t)$ in the aggregate output process (2).

The uncertainty in the model results from two risk sources: the market risk factor and the undiversifiable LNIS. The market factor is captured by fluctuations $dZ(t)$ in market/economic conditions. The undiversifiable disastrous income shock is captured by the Poisson shock $N(t)$ with a small probability that comes as the LNIS.

The Optimal Consumption and Investment Problem. The agent's optimal consumption and investment problem with the LNIS is to maximize over the infinite horizon her constant relative risk aversion (CRRA) utility from intermediate consumption with wealth constraints (3) and (4) by optimally controlling her consumption c and investment π.[17]

[15] For more details about this dynamic wealth constraint, refer to an online appendix.

[16] This wealth constraint is same with the free borrowing against wages (Dybvig and Liu, 2010). In other words, the agent is allowed to borrow against the present value of her future wages (or the human capital).

[17] Throughout the paper, we only consider the set of admissible policies of consumption $c(t)$ and investment $\pi(t)$.

When $\delta = 0$ ($\tau = \infty$), i.e., without the LNIS the aggregate earnings are $\xi I(t)$, where aggregate output $I(t)$ follows a geometric Brownian motion:

$$dI(t) = \mu^I I(t)dt + (\sigma^I)^\top I(t)dZ(t), \quad I(0) = I > 0. \tag{5}$$

The value function is then

$$V^B(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}\Big] = K\frac{(w + \xi I/\beta_1)^{1-\gamma}}{1-\gamma}, \tag{6}$$

where $\beta > 0$ is the subjective discount rate, $c(\cdot)$ denotes reduced-form consumption of goods and services, $\gamma > 0$ is the constant coefficient for the agent's relative risk aversion, and

$$K = A^{-\gamma},\ A = \frac{\gamma - 1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma},\ \theta = (\sigma^\top)^{-1}(\mu - r\mathbf{1}).$$

Here, θ is the Sharpe ratio vector.

When $\delta = \infty$ ($\tau = 0$), i.e., in the LNIS after the Poisson shock $N(t)$, the aggregate earnings are $k\xi I(t)$, where aggregate output is the same as in (5). The value function is then

$$V^A(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta(t-\tau)}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big] = K\frac{(w + k\xi I/\beta_1)^{1-\gamma}}{1-\gamma}. \tag{7}$$

When $0 < \delta < \infty$ ($0 < \tau < \infty$), i.e., with the LNIS, the value function is to maximize the CRRA utility from consumption before the LNIS ($0 < t < \tau$) and after the LNIS ($\tau \le t < \infty$) as the following:

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big], \tag{8}$$

which is subject to the wealth constraints (3) and (4). The value function $V(w, I)$ reduces to either $V^B(w, I)$ or $V^A(w, I)$ in the two limiting cases of either $\delta = 0$ ($\tau = \infty$) or $\delta = \infty$ ($\tau = 0$) as follows:

$$V(w, I) = \begin{cases} V^B(w, I) & \text{without the LNIS, i.e., when } \delta = 0\ (\tau = \infty), \\ V^A(w, I) & \text{in the LNIS, i.e., when } \delta = \infty\ (\tau = 0). \end{cases} \tag{9}$$

In these limiting cases, the agent's problem reduces to Friedman (1957) and Merton (1969, 1971) in that her optimal consumption strategy follows Friedman's PIH and her optimal investment strategy Merton's investment rule (Theorem 3.1). Specifically, without the LNIS ($\tau = \infty$), the optimal consumption follows Friedman's (1957) PIH

$$c(t) = \hat{A}\Big(w + \frac{\xi I}{\beta_1}\Big),$$

where

$$\hat{A} = \frac{\gamma - 1}{\gamma}\Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},$$
$$\beta_2 = \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$
$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

and the optimal investment follows Merton's (1969, 1971) investment rule

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1}.$$

In the LNIS ($\tau = 0$), the optimal consumption and investment strategies follow by replacing aggregate earnings ξI with $k\xi I$ in Friedman's (1957) PIH and Merton's (1969, 1971) investment rule stated above.

3 Generalized Permanent Income Hypothesis

General Case. By the principle of dynamic programming the value function given in (8) can be rewritten as the following:

$$V(w, I) = \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(W(\tau), I(\tau))\Big].$$

This paper generalizes the two special cases (Friedman's PIH and Merton's investment rule) with the LNIS in the empirically plausible range of $0 < \delta < \infty$ ($0 < \tau < \infty$). Integrating out the random arrival τ of the LNIS in (8) allows us to restate the optimal consumption and investment problem as the following:[18]

$$V(w, I) = \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \qquad (10)$$

[18]For the details of the derivation, refer to an online appendix.

subject to (3). The last term involving intensity δ of the LNIS on the right-hand side of (10) represents the agent's maximized utility value in the aftermath of the LNIS. The agent's utility value in the LNIS significantly interacts with the utility value before the LNIS, thereby resulting in qualitative and quantitative adjustments in the optimal strategies with the LNIS in the ultimate attainment of the totally maximized utility value given by (10). Put differently, the agent with the LNIS becomes forward looking and aims to maximize not only her utility value before the LNIS, but interestingly also her utility value after the LNIS.

We also find that the LNIS makes the maximized expected discounted utility $-\infty$ (or the maximized expected utility $+\infty$) if the agent would choose to follow Friedman's PIH and Merton's investment policy. For instance, according to the PIH, the agent could be able to finance her consumption needs by borrowing against her human capital. In this case, however, the last term involving intensity δ of the LNIS on the right-hand side of (10) would be highly likely to be $-\infty$, especially as wealth approaches the borrowing limit $-\xi I(t)/\beta_1$ given by (4).[19] Therefore, developing the gereralized PIH framework with the LNIS should address this challenge of $-\infty$.

In an attempt to resolve such an infinity issue caused by the LNIS, we consider a catastrophically low time-varying value of wealth as a new borrowing limit instead of (4) which is reminiscent of a starvation level below which the agent cannot sustain herself financially and thus, do not invest in the stock market at all. The new borrowing limit is then given by

$$W(t) > -L(t) > -\frac{k\xi I(t)}{\beta_1}, \quad \text{for all } t \geq 0, \tag{11}$$

where $L(t)$ is a given nonnegative time-varying function. With the time-varying function $L(t)$, borrowing against human capital is now constrained fully or partly. Thus, the extent to which credit is tightened, i.e., the level of lower bound of wealth becomes a real consideration.

[19] Such an infinity issue has been also acknowledge by Bensoussan *et al.* (2016).

The Optimal Savings. We derive in closed-form the optimal consumption and investment strategies of the agent with the LNIS, which result in an explicit expression for the optimal (riskless) savings.

Theorem 3.1 *The optimal consumption strategy* $c(t)$ *and the optimal investment strategy* $\pi(t)$ *of the agent with the LNIS are derived in closed-form:*

$$c(t) = (\hat{A} + \delta)\Big(w + \frac{\xi I}{\beta_1} - \xi I B_\delta^* z^{-\alpha_\delta^*} - IP\Big), \tag{12}$$

$$\begin{aligned} \pi(t) = & \frac{1}{\gamma}\sigma^{-1}\theta w \\ & + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*} \\ & - \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c(t)^{-\gamma} + (\gamma\alpha_\delta - 1) \times IP1 + (\gamma\alpha_\delta^* - 1) \times IP2\Big], \end{aligned} \tag{13}$$

where

$$\hat{A} = \frac{\gamma - 1}{\gamma}\Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},$$

B_δ^* *and* $\overline{z}$ *are the two constants to be determined by the boundary conditions*

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \textit{and} \quad G'(\overline{z}) = 0,$$

$\alpha_\delta > 1$ *and* $-1 < \alpha_\delta^* < 0$ *are the two roots to the following characteristic equation:*

$$F(\alpha; \delta) \equiv -\frac{1}{2}||\beta_3||^2\alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0,$$

$G(z)$ *is a dual function of the value function* $V(w, I)$ *and it is given by:*

$$\begin{aligned} G(z) = & \frac{1}{\hat{A} + \delta} z^{-1/\gamma} + B_\delta^* z^{-\alpha_\delta^*} \\ & + \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta - 1)z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ & + (\alpha_\delta^* - 1)z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big], \end{aligned} \tag{14}$$

with

$$\beta_2 = \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$

$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

IP represents the integral parts of LNIS-induced precautionary savings and it is given by

$$IP = IP1 + IP2,$$

$$IP1 = \frac{2\delta K(\alpha_\delta - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu < 0,$$

$$IP2 = \frac{2\delta K(\alpha_\delta^* - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu > 0.$$

Friedman's (1957) Consumption. Without the time-varying borrowing constraints (11) and the LNIS, i.e., when $B_0^* = 0$ and $\delta = 0$, the agent's optimal consumption strategy (12) can be rewritten by the following Friedman's (1957) PIH:

$$c(t) = \hat{A}\Big(w + \frac{\xi I}{\beta_1}\Big),$$

which means that the agent's consumption can be annuitized from her total available resources. Further, the marginal propensity to consume out of financial wealth is constant implying that regardless of wealth levels, the agent's optimal consumption to total wealth ratio is well maintained at constant rate.

Merton's (1969, 1971) Investment. The classic Merton (1969, 1971) investment rule can be revisited:

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I \frac{\xi I}{\beta_1}, \tag{15}$$

which comes from (13) when $\sigma_I > 0$ and $\delta = 0$, i.e., with output uncertainty but without the LNIS. The first term on the right hand side of (15) represents the mean-variance asset allocation and the second one represents the demand for hedging (or the intertemporal hedging component) against the output uncertainty.

Theorem 3.1 allows us to obtain the resulting optimal (riskless) savings by measuring the wedge between total wealth (financial wealth+human capital) and the sum of consumption and investment. Specifically, we identify and quantify three different optimal savings motives in the following Corollary: (i) PIH-implied optimal savings, (ii) Borrowing-constraints-induced optimal savings, and (iii) LNIS-induced optimal savings.

Corollary 3.1 *We identify and quantify three different optimal savings motives as follows.*

(i) PIH-implied optimal savings

$$\equiv \Big(w + \frac{\xi I}{\beta_1}\Big) - c(t; B_0^* = 0, \delta = 0) - \pi(t; B_0^* = 0, \delta = 0)$$

$$= \Big(1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta\Big)\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^{I}\frac{\xi I}{\beta_1}.$$

(ii) Borrowing-constraints-induced optimal savings

$$\equiv \Big(w + \frac{\xi I}{\beta_1}\Big) - c(t; \delta = 0) - \pi(t; \delta = 0)$$

$$= \textit{PIH-implied optimal savings}$$

$$+ \Big(\hat{A} - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^{I})(\gamma\alpha_0^* - 1)\Big)\xi I B_0^* z^{-\alpha_0^*}.$$

(iii) LNIS-induced optimal savings

$$\equiv \Big(w + \frac{\xi I}{\beta_1}\Big) - c(t) - \pi(t)$$

$$= \textit{PIH-implied optimal savings} + \textit{LNIS-PS},$$

where the LNIS-induced precautionary savings (LNIS-PS) are given by

$$\begin{aligned} \textit{LNIS-PS} = &-\delta\Big(w + \frac{\xi I}{\beta_1}\Big) + \Big(\hat{A} + \delta - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^{I})(\gamma\alpha_\delta^* - 1)\Big)\xi I B_\delta^* z^{-\alpha_\delta^*} \\ &+ \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^{I})\frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c(t)^{-\gamma} \\ &+ \Big(\hat{A} + \delta - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^{I})(\gamma\alpha_\delta - 1)\Big) \times \textit{IP1} \\ &+ \Big(\hat{A} + \delta - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^{I})(\gamma\alpha_\delta^* - 1)\Big) \times \textit{IP2}. \end{aligned} \tag{16}$$

The PIH-implied optimal savings show that the marginal propensity to save (MPS) out of financial wealth is $1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta$, which implies that with respect to one unit increase of wealth the constant portion of the agent's extra money aside from consumption portion $\hat{A}$ and investment portion $\frac{1}{\gamma}\sigma^{-1}\theta$ is to be optimally put into her riskless savings. This savings strategy, however, has been at odds with empirical evidence (Federal Reserve report, 2017; EU-SILC 2017) in that the agent's savings are too small to address the financial challenges on her future consumption.

In addition to the PIH-implied optimal savings, the extra terms on the right hand side of borrowing-constraints-induced optimal savings in 3.1 represent additional precautionary savings motive for avoiding being binded by the borrowing constraints given in (11).

The smaller MPS with borrowing constraints implies that with respect to one unit decrease of wealth the agent is inclined to less reduce her savings amount, as she is responsible for maintaining her wealth to be larger than the time-varying constraint $-L(t)$ given in (11) in all states. Interestingly, the MPS further decreases as wealth decumulates (as a result, z becomes larger). This shows that the borrowing-constrained precautionary savings motive has a progressively more stronger impact on the agent's total savings when wealth is small, thereby further increasing demand for savings at low levels of wealth in the preparation against market downturns.

Bewley (1977) and Campbell (1987) point out that an income shock hardly affects the optimal savings decision of wealthy people because they are already well prepared for meeting their future consumption needs by relying on their enough wealth. Contrary to the theoretical predictions of Bewley (1977) and Campbell (1987), many empirical studies show that positve and even high savings rates are very common amongst wealthy people (Dynan *et al.*, 2004; Buera, 2009; De Nardi *et al.*, 2010; De Nardi and Fella, 2017). As complementary to these explanations for the high savings rates of wealthy people, we particularly emphasize that the agent who is at the higher end of wealth could have a savings demand for precautionary reasons in the event of the LNIS. Indeed, the integral parts $IP1$ and $IP2$ of $LNIS - PS$ can play a role to further decrease the MPS and thus, the agent tends to less reduce her savings amount with respect to one unit decrease of wealth. Rather, the agent reduces her consumption amount by $(\hat{A} + \delta) \times IP$ as in (12) and increases her savings amount as in the LNIS-induced optimal savings (or in the LNIS-induced precautionary savings) in Corollary 3.1.

Given the differences between the PIH-implied optimal savings, borrowing-constraints-induced optimal savings, and LNIS-induced optimal savings as we have analyzed so far, it is worth to thoroughly investigate a role of the LNIS in the following two points: (i) what and how the extra LNIS-induced precautionary savings motive would affect the general equilibrium interest rate, and (ii) such a savings movie could improve the equilibrium model's ability to match the equity premium and risk-free rate observed from the data.

4 General Equilibrium Analysis

Equilibrium Building Blocks. We consider a simple pure exchange economy in the type of Lucas (1978). The economy is populated by a representative agent who encounters the LNIS. The agent is entitled to an aggregate endowment to be consumed in equilibrium and she is assumed to trade a riskless bond and multiple risky stocks distributing the dividend. The returns to these assets adjust to represent a no-trade equilibrium.

Definition 4.1 *An equilibrium can be characterized as a collection of* (r, μ, σ) *and optimal strategies* $(c(t), \pi(t))$ *such that the consumption good, stock, and bond markets clear as*

$$
\begin{aligned}
&c(t) = I(t),\\
&\pi^j(t) = S^j(t)W(t), \quad j = 1, ..., N,\\
&W(t) = \sum_{j=1}^{N} S^j(t),
\end{aligned}
$$

where N *is the number of risky stocks.*

State Price Density. The following theorem characterizes the unique state price density (or the shadow price) in the presence of income risk including the LNIS.

Theorem 4.1 *The unique state price density is derived in closed-form:*

$$\xi^{\hat{\delta}}(t) = \exp\left\{ \ln\left(\frac{\hat{\delta}}{\delta}\right) \boldsymbol{1}_{\{\tau \le t\}} - (\hat{\delta} - \delta)t \right\} H(t), \tag{17}$$

where

$$\hat{\delta} = \left(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\right)^{-\gamma} \frac{\delta K}{z},$$

τ *is the arrival time of a Poisson shock,* $\boldsymbol{1}$ *is an indicator function that gives* 1 *if the Poisson shock occurs at time* t *and* 0 *otherwise and the dynamics of* $H(t)$ *are given by*

$$dH(t) = -H(t)\{rdt + \theta^{\top} dZ(t)\}, \quad H(0) = 1.$$

The unique state price density given in (17) is a generalized version of the well-known Arrow-Debreu price. The identified quantity $\xi^{\hat{\delta}}(t,\zeta)$ can be regarded as the Arrow-Debreu price per unit probability P of one unit consumption good in state $\zeta \in \Omega$ at time t.[20]

Theorem 4.1 allows a convenient multiplicative separation of the traditional Arrow-Debreu price and the LNIS adjustments. In the absence of the LNIS ($\delta = \hat{\delta}$), the state price density (17) reduces to the conventional Arrow-Debrew price, and presents only the output uncertainty adjustments. In the presence of the LNIS ($\delta \neq \hat{\delta}$), the extra income shock adjustments further increase the canonical Arrow-Debrew price and lead to the generalized Arrow-Debreu price $\xi^{\hat{\delta}}(t)$ in (17).

Equilibrium Consumption Price. The rational expectations equilibrium consumption price (equilibrium state price density) must satisfy the Euler equation (Stokey and Lucas, 1989).[21] When solving the optimal consumption and investment problem of (10) with dynamic wealth constraint (3) and lower bound of wealth (11), we can derive the following Euler equation:

$$U'(c(t)) = \lambda e^{(\beta-(\hat{\delta}-\delta))t} H(t).$$

Using Itô's formula, we explicitly derive the following equilibrium consumption dynamics:[22] for $t < \tau$,

$$\frac{dc(t)}{c(t)} = -\frac{U'(c(t))}{U''(c(t))c(t)} \Big[\Big\{ \Big(r + (\hat{\delta}-\delta) \Big) - \beta \Big\} dt - \theta^{\top} dZ(t) \Big].$$

Therefore, the Euler equation shows that equilibrium consumption growth is measured as a gap between income-risk-adjusted interest rate of $(\hat{\delta}-\delta)$ and subjective discount rate of β, weighted by the elasticity of intertemporal substitution. The risk-neutral Poisson

[20] Basically, the Arrow-Debrew price is the future price of one unit consumption good. It can serve as a shadow price for discounting future costs and benefits in financial analysis. The derived unique state price density determines the risk-neutrality with respect to the LNIS. This price can be used as the Randon-Nikodym derivative for measure change purposes.

[21] An Euler equation is a differential equation that represents an intertemporal first-order condition for optimal consumption (Durlauf and Blume (2008), pp. 1854-1855).

[22] This dynamics are equivalent to the dynamics of equilibrium consumption price $e^{-\beta t}\xi^{\hat{\delta}(t)}$ for $t < \tau$.

intensity $\hat{\delta}$ is included in addition to the original Poisson intensity δ. $\hat{\delta}$ either increases or decreases the equilibrium consumption growth relative to risk-free interest rate r by the amount of $\hat{\delta} - \delta$. When $\hat{\delta} = \delta$, the LNIS can be fully diversified; as a result, it is rewarded with the zero risk premium. However, when $\hat{\delta} \neq \delta$, the LNIS cannot be diversified, so it should be compensated for by the nonzero risk premium $\hat{\delta} - \delta$.[23] In the context of the traditional risk-return trade-off (i.e., high risk and high return), $\hat{\delta}$ should be larger than δ,[24] as a result, the income risk premium should be positive. Thus, the effective risk-adjusted interest rate represents such a risk compensation for additional exposure to the LNIS, and thereby increases both equilibrium consumption growth and equilibrium consumption price compared to the case without the LNIS.[25] This would imply that the amount of present consumption the agent would be willing to give up now to receive one more unit of future consumption becomes larger than without the LNIS.

Equilibrium Risk-Free Interest Rate. The following theorem solves the equilibrium risk-free interest rate.

Theorem 4.2 *The equilibrium risk-free interest rate is derived in closed-form:*

$$r = \beta + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)(\sigma^I)^2 - (\hat{\delta}(r) - \delta), \tag{18}$$

where μ^I and σ^I represent the expected consumption growth rate and volatility of consumption growth rate, and the constant $\hat{\delta}(r)$ is determined by solving the following non-linear algebraic equation:

$$\hat{\delta}(r) = \Big\{\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)\Big/\Big(\frac{w}{\xi I} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)\Big\}^{\gamma}\{\beta_1(\hat{\delta}(r))\}^{\gamma}\delta K(r)$$

[23] The canonical CAPM does not generate the nonzero income risk premium as we obtained, whereas in this paper, the LNIS can be thought of as extra undiversifiable risk source, accordingly, it should be priced.

[24] This result can be also confirmed with a wide range of parameter values.

[25] Intuitively, the cost of one unit of equilibrium consumption increases with increase in a person's uncertainty about her future earnings. Further, the presence of the LNIS may increase the positive skew of the distribution of equilibrium consumption price (or equilibrium state price density), and this change will affect existing frameworks for asset pricing and risk management.

with

$$\beta_1(\hat{\delta}(r)) = \beta + (\gamma - 1)\mu^I - \frac{1}{2}\gamma(\gamma-1)(\sigma^I)^2 - (\hat{\delta}(r) - \delta),$$
$$K(r) = \Big\{\frac{\gamma-1}{\gamma}\Big(r + \frac{\gamma(\sigma^I)^2}{2}\Big) + \frac{\beta}{\gamma}\Big\}^{-\gamma}.$$

Our equilibrium results with the LNIS on the risk-free interest rate sit somewhat easily with the key role of the savings inequality in shaping the wealth concentration among the rich (Piketty, 2014; Acemoglu and Robison, 2015). In the existing equilibrium literature without the LNIS, the equilibrium difference of $\beta - r$ is large, so higher-wealth individuals have a tendency to consume more and save less. This is because their financial wealth grows at the rate of β, whereas their savings grow at the rate of r. When financial wealth is disproportionately concentrated on the top of the wealth distribution, i.e., with the unequally distributed wealth, the rich dis-save in equilibrium. Thus, the low or nearly zero wedge of $\beta - r$ is the key to generate high savings rates for the wealthy.

How do we obtain such a low difference of $\beta - r$ in equilibrium? The possibility is to understand in equilibrium the effects of the LNIS on the interest rate. The representative agent facing the LNIS demands extra premium for her risk exposure to the LNIS, which corresponds to a decrease in β by the amount of the risk premium $\hat{\delta} - \delta$ depending upon Poisson arrival intensity δ of the LNIS.

This effect of δ on interest rates is particularly relevant to today's low interest-rate situation, in that the LNIS has placed a heavy burden on the choice of equilibrium interest rate. This additional burden also leads to a reduction of interest rate. A wide range of private and public insurance would greatly improve security of earnings in the long run. Thus, strategies and solutions focusing on the elimination of risk potentially catastrophic individual earnings losses as much as possible would help to reduce the amount of compensation for exposure to the LNIS, and might return the interest rate to its normal status during times of stable economy.

5 Quantitative Analysis

In this section, we perform quantitative analysis to illustrate the optimal strategies to attain consumption smoothing when the LNIS can occur.

5.1 Parameter Values

Financial Market. We consider only two assets in the financial market: a riskless bond and a risky stock. We choose equity premium, $\mu - r$, as 4% and risk-free rate, r, as 2%. The stock volatility, σ, is assumed to be 20%.[26]

People Preferences. The coefficient of relative risk aversion, γ, is set to 2. We adopt the common value of 4% for the subjective discount rate, β. The mortality rate, ν, is fixed to 2%, i.e., the expected time to death is 50 years.

Aggregate Earnings. For simplicity, aggregate earnings are assumed to be given by a constant income stream, $\epsilon \equiv \xi I$, over the life cycle. When a jump-type LNIS occurs, aggregate earnings decrease from ϵ to $k\epsilon$, $k \in (0, 1)$. For prototype example, we let the LNIS be associated with the risk of forced unemployment; this risk can be also a leading cause of the credit tightening formulated by (19).[27]

Our choice of Poisson intensity $\delta = 1\%$ is very conservative compared to Wang et al. (2016) who have chosen the arrival rate of large discrete (jump) earnings shocks as 5%. The recovery parameter k is set to 40%.[28]

[26]Compared to the century-long sample (1891-1994) by Campbell (1999), the risk-free rate of 2% is reasonable, but the equity premium of 4% is somewhat conservative. In our general equilibrium analysis, the risk-free rate and equity premium can be determined to 2% and 4%, respectively, with the reasonable values of risk aversion.

[27]A key insight comes from economic recessions followed by human capital depreciation during long-term periods of unemployment. During the 2007-2009 Great Recession in the United States, many people have experienced the unprecedented largest reductions in their consumption and unemployment.

[28]In practice, U.S. households have been rescued by a safety net against the aftermath of the LNIS

Lower Bound of Wealth. The lower bound $-L(t)$ with $L(0) = L$ of wealth in (11) can be empirically plausible using the following relationship:[29]

$$L = \omega \frac{r + \nu + \delta k}{(r + \nu + \delta)(r + \nu)} \epsilon, \quad \text{for} \quad 0 \leq \omega < 1, \tag{19}$$

where ω represents the extent to which credit is tightened and $\nu > 0$ is the agent's constant mortality intensity when the time to death is distributed with an exponential distribution.[30] The utility related to death is normalized as zero.[31] For numerical illustrations, we do not allow for borrowing against the present value of future income, i.e., $\omega = 0$.[32]

5.2 Numerical Illustrations

In this section, we obtain numerical solutions and present graphical illustrations to provide some details to the discussion about the general equilibrium quantities with the LNIS.

Equilibrium Interest Rate. In the absence of the LNIS ($\delta = 0$), the equilibrium interest rate is 6.55%, but in the presence of the LNIS ($\delta > 0$), it drops significantly (Figure 1). This relationship implies the important discontinuity and dramatic change in the interest rate even when δ is very small small. For instance, the equilibrium interest rate decreases 41.53% (i.e., to 3.83%) as δ increases from 0 to 2%.

(for example, possibly caused by forced unemployment), and recover 20% of the income that they earned before unemployment (Carroll *et al.*, 2003).

[29]The relationship shows an accurate reflection of human capital adjusted by the LNIS. The derivation details are available in an online appendix.

[30]The constant mortality rate assumption is made for parsimony of the model, helping explore horizon-dependent polices in the simplest possible economic environment. The derived model predictions are consistent with the typical life-cycle advice. A more realistic model would allow for a Gompertz force of mortality, which is quite relevant to the actuarial literature.

[31]On account of this normalization, we do not consider motive for bequest. The presence of bequest motive is expected to reinforce the negative impacts of the LNIS.

[32]For the effects of credit tightening with a range of values for ω on the optimal strategies, refer to further numerical results in an online appendix.

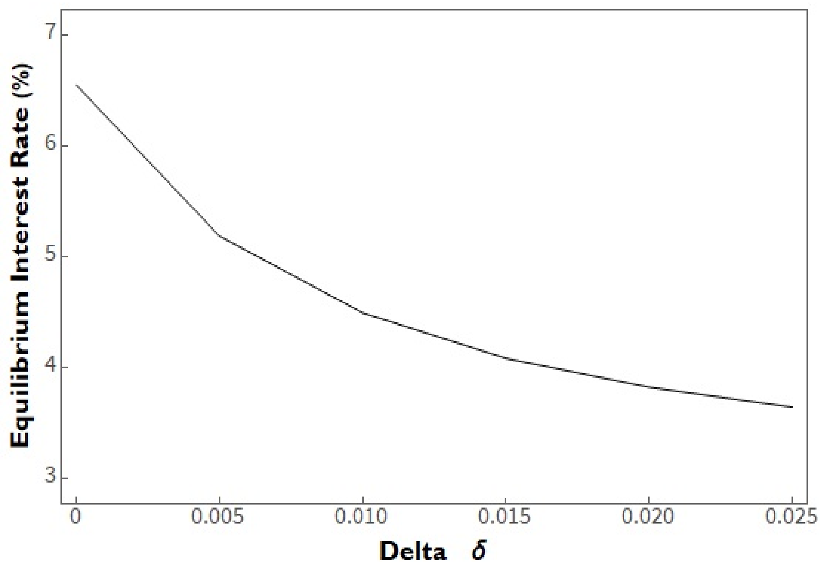

Figure 1: **Equilibrium interest rates**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 2$ (risk aversion), $\mu^I = 1.52\%$ (expected consumption growth rate), $\sigma^I = 4.06\%$ (volatility of consumption growth rate), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), $k = 40\%$ (recovery rate), and $\omega = 0$ (borrowing constraint). For the expected consumption growth rate and volatility of consumption growth rate, μ^I and σ^I, we have used the Robert J. Shiller's real monthly dividend data from 1926 to 2016 in "Irrational Exuberance" published by Princeton University Press. Note: In the absence of the LNIS ($\delta = 0$), the equilibrium interest rate is 6.55%, but in the presence of the LNIS ($\delta > 0$), it drops significantly, implying the important discontinuity and dramatic change in the interest rate even when δ is small.

Risk aversion γ also affects the equilibrium risk-free interest rates (Figure 2). When $\delta > 0$, high values of γ no longer counterfactually generate high risk-free interest rates, so the risk-free rate puzzle (Weil, 1990) is avoided. Rather, an increase in risk aversion can lead to a decrease in risk-free rate in the presence of the LNIS.

The LNIS includes a low-probability, depression-like third state of Rietz (1988)'s model in the individual's income process, which can be regarded as a different application of the rare disaster risk hypothesis by Rietz (1988).[33] Consistent with this hypothesis, the possibility of the LNIS can account for high risk premium on bonds but through the

[33] The rate disaster hypothesis arguably states that the slim chance of rare disasters (e.g., economic crisis or war) can dominate the determinaiton of asset risk premia. The seminal work of Rietz (1988), Barro (2006) and Gabaix (2012) have established different versions of the rate disaster hypothesis, thereby explaining empirical regularities, such as the equity premium puzzle and the risk-free rate puzzle.

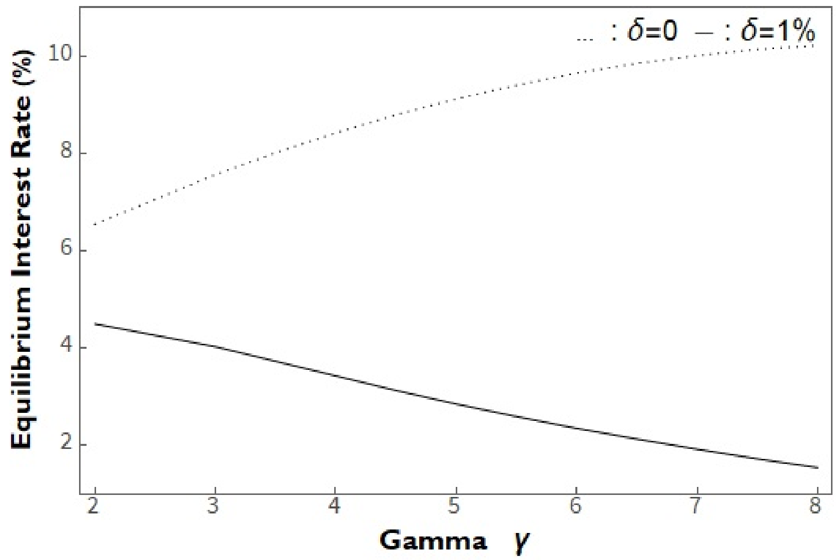

Figure 2: **Equilibrium interest rates**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 2$ (risk aversion), $\mu^I = 1.52\%$ (expected consumption growth rate), $\sigma^I = 4.06\%$ (volatility of consumption growth rate), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), $k = 40\%$ (recovery rate), and $\omega = 0$ (borrowing constraint). Note: When $\delta > 0$, high values of γ no longer counterfactually generage high risk-free interest rates, so the risk-free rate puzzle (Weil, 1990) is avoided. Rather, an increase in risk aversion can lead to a decrease in risk-free rate in the presence of the LNIS.

precautionary savings channel. With the LNIS, people's demand for precautionary savings is sufficiently strong making her save at a high rate and lowering the equilibrium interest rate significantly.

The presence of the LNIS drives down the risk-free interest rate by stimulating the precautionary savings mechanism, thereby maintaining the risk-free rate low. Similar to Bewley (1977) and Campbell (1987), we could confirm the amount of optimal savings decreases in financial wealth w (Figure 3).[34] This is because consumption of the wealthy

[34]The downward-sloping part of individual savings profile can be understood within the traditional life-cycle framework. In the context of the life-cycle hypothesis, the savings should have the downward-sloping part at the high end of wealth (Jappelli, 1999; Deaton and Paxson, 2000; Attanasio and Szekely, 2000; Cocco *et al.*, 2005; Benzoni *et al.*, 2007). Empirically, the downward-sloping part has been observed by Jappelli (1999) using data from Italy, by Deaton and Paxson (2000) using data from Taiwan and Thailand, and by Attanasio and Szekely (2000) using data from East Asia and Latin America. These empirical observations have been justified by theoretical life-cycle models such as Cocco *et al.* (2005) and

can be financed by wealth, without resorting to income. Intuitively, the ability to self-insure against the income shock improves when wealth is large, so the optimal savings decrease as wealth increases.[35]

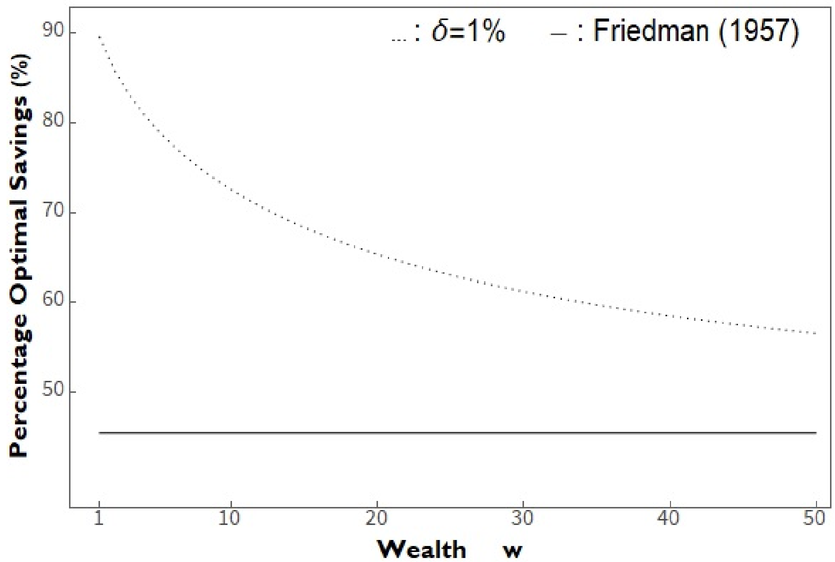

Figure 3: **Precautionary savings amount.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), $k = 40\%$ (recovery rate), and $\omega = 0$ (borrowing constraint). Note: The amount of LNIS-induced optimal savings ($\delta = 1\%$) is significantly larger than the Friedman's case without the LNIS ($\delta = 0$).

Although an income shock does not seem to significantly affect the optimal savings of people who are at the higher end of wealth (Bewley, 1977; Campbell, 1987), the LNIS could affect substantially the savings rate of rich people.[36] Indeed, the optimal savings are still

Benzoni *et al.* (2007).

[35]The income-to-wealth-ratio is inversely related to individual wealth. The income is a major staple of the relatively low-wealth people and they should concern themselves with diversifying the negative effects of the LNIS, thereby continuing being able to afford what they can currently afford by saving less. Relative to the low-wealth people, the income is a smaller staple of the high-wealth people, so they have greater tolerance for risk than the low-wealth people. Hence, the high-wealth people would rather not be concerned with diversification. They can afford to pay further for economic preparedness and emergency savings, thereby saving more to meet the required precautionary savings for consumption smoothing and thus, being well-prepared for unexpected hardship caused by the LNIS.

[36]Empirically, the average ratio of wealth-to-income between 1952 and 2016 in the U.S. was 6.6. Since

very high for people who are at the higher end of wealth. This result may partially explain positive and even high savings rates amongst wealthy people which have been observed by empirical and anecdotal evidence (Dynan *et al.*, 2004; Buera, 2009; De Nardi *et al.*, 2010; De Nardi and Fella, 2017).[37] The reason is that people would be better off keeping accumulating extra wealth for precautionary reasons in the event of the LNIS, and use this extra wealth as a buffer against the LNIS. Further, an unexpected, exogenous, and permanent jump-type LNIS is much harder to buffer than adverse diffusive-type income shocks, so readiness for the LNIS requires a large amount of savings.

Matching Equity Premium and Risk-Free Rate. Using the LNIS, our equilibrium results are matched up with the observed risk-free rate and equity premium. We have tried to match our model with the century-long sample from 1891 to 1994 by Campbell (1999) and the long historical sample from 1871 to 2011 by the website of Robert Shiller (http://www.econ.yale.edu/∼shiller/data/chap26.xls) (Table 1). The presence of the LNIS dramatically improves the model's ability to match asset prices. The standard asset pricing framework by Lucas (1978) without the LNIS requires negative values for the subjective discount rate which are not empirically plausible in order to obtain the risk-free rate of 1.74% (1.48%) which is lower than the observed rate of 1.96% (2.8%) from the century-long sample (the long historical sample). While the risk-free rate of 1.96% (2.8%) generated by our framework with the LNIS is exactly the same with that observed from the century-long sample (the long historical sample), thereby requiring empirically plausible parameters as follows: LNIS shock $\delta = 1\%$ ($\delta = 4.5\%$), subjective discount rate $\beta = 1.65\%$ ($\beta = 1\%$),

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labor income is normalized as one in our paper, high-wealth people can be regarded as those having wealth more than 6.6.

[37] The recent studies offer many alternative explanations for the upward-sloping part of individual savings profile: a positive relation between savings rates and income (Dynan *et al.*, 2004), entrepreneurship purposes for entering and expanding business (Buera, 2009), out-of-pocket medical expenses patterns (De Nardi *et al.*, 2010), the mix of bequests and human capital, entrepreneurship, and medical-expense risk (De Nardi and Fella, 2017). The explanation we provide here, while different, would be regarded as complementary to these.

Estimated consumption and return parameters	1891-1994	1871-2011
Expected consumption growth rate μ^I	1.74%	2.3%
Consumption volatility σ^I	3.26%	3.3%
Stock Volatility σ	18.53%	18.2%
Risk-free rate r	1.96%	2.8%
Equity premium $\mu - r$	6.26%	5.2%

(a) Data

Required parameters	1891-1994		1871-2011	
LNIS δ	0	1%	0	4.5%
Discount rate β	-9.81%	1.65%	-15.53%	1%
Risk aversion γ	10	10	10	9
Model-generated equilibrium quantities	**1891-1994**		**1871-2011**	
Risk-free rate	1.74%	1.96%	1.48%	2.8%
Equity premium	6.04%	6.04%	6.0%	5.4%

(b) Model results with required parameters

Table 1: Table (a) reports the annualized parameter values for consumption and return for the century-long sample (1891-1994) by Campbell (1999) and the long historical sample (1871-2011) by the website of Robert Shiller (http://www.econ.yale.edu/∼shiller/data/chap26.xls). Table (b) reports a comparison of the model-generated equilibrium results from the model without the LNIS ($\delta = 0$) and the model with the LNIS ($\delta > 0$).

risk aversion $\gamma = 10$ ($\gamma = 9$).[38]

The intuitive interpretation of our improved ability to match the observed asset prices is that the LNIS faced by individuals causes them likely to reduce consumption very much to secure extra reserves in preparation for financing future consumption needs by using their savings. Such a savings decision encourages reduction in equity demand, so the equity premium increases and the risk-free rate instead decreases due to their high savings.

Table 5.2 reports the estimated consumption and return parameter values from four sample periods: 1889-1978 (Lucas, 1994), 1890-1997 (Gomes and Michaelides, 2008), 1929-2009 (Constantinides and Ghosh, 2017), and 1930-2008 (Schmidt, 2016). To investigate the ability of our model with the LNIS to match the risk-free rate and equity premium from

[38]According to Mehra and Prescott (1985), the upper bound of risk aversion is known as 10.

Estimated consumption and return parameters	1889-1978	1890-1997	1929-2009	1930-2008
Expected consumption growth rate μ^I	1.8%	1.7%	2.0%	1.93%
Consumption volatility σ^I	3.7%	3.3%	2.0%	2.16%
Stock Volatility σ	16.7%	19.81%	18.7%	20.28%
Risk-free rate r	1.0%	1.58%	0.6%	0.57%
Equity premium $\mu - r$	6.0%	6.74%	7.0%	7.09%

Table 2: Table reports the estimated consumption and return parameter values from sample periods: 1889-1978 (Lucas, 1994), 1890-1997 (Gomes and Michaelides, 2008), 1929-2009 (Constantinides and Ghosh, 2017), and 1930-2008 (Schmidt, 2016).

the data (Table 5.2), we compare our model with four representative general equilibrium models considering unhedgeable income risk by Lucas (Lucas, 1994), GM (Gomes and Michaelides, 2008), CG (Constantinides and Ghosh, 2017), and Schmidt (Schmidt, 2016) (Table 5.2). Table 5.2 (a), (b), (c), (d) show the comparison results between ours and Lucas, ours and GM, ours and CG, ours and Schmidt, respectively. The second last column of each table reports the number of model parameters used to match asset prices. The details are as follows:

- Lucas (1994) uses 5 model parameters: discount rate β, risk aversion γ, idiosyncratic shocks, short sale constraint, borrowing constraint

- Gomes and Michaelides (2008) use 6 model parameters: discount rate β, risk aversion γ, two EIS (ψ) parameters for two different type agents, deviation of productivity shock, standard deviation of income shock

- Constantinides and Ghosh (2017) use 14 model parameters: 3 preference parameters of discount rate β, risk aversion γ, EIS (ψ), 3 parameters for income shocks, 2 parameters of mean and volatility of aggregate consumption growth, 3 parameters for state variable dynamics, 3 parameters governing aggregate dividend growth dynamics

- Schmidt (2016) uses 21 model parameters[39] including 3 preference parameters of discount rate β, risk aversion γ, EIS (ψ) and 4 parameters for consumption and

[39]Refer to Table 3 Summary of Parameters for the Quantitative Model in Schmidt (2016).

income shocks such as income shock intensity δ

- Our model uses 4 model parameters: discount rate β, risk aversion γ, LNIS δ, recovery k

The last column of each table reports the mean square error(MSE) which is the average squared difference between the observed risk-free rate and equity premium (Data) and the model-generated rates (Models). The last row of each table reports the optimized our model results to match asset prices by minimizing the sum of the squared relative errors between the historical rates and our model-generated rates.

The comparison of ours and Lucas highlights the incremental contribution to the ability to match asset prices from 1889-1978, where Lucas argues that idiosyncratic shocks to income do not matter for asset prices. As compared with 9.3% and 0.7% generated by Lucas for the risk-free rate and equity premium, respectively, ours generates (with the same discount rate, a slightly higher risk aversion, and the empirically plausible LNIS $\delta = 5.0\%$) the risk-free rate of 3.8% and the equity premium of 1.9% (Table 5.2 (a)).

The comparison of our model and GM also emphasizes the ability of the LNIS to match asset prices from 1890-1997, where GM arguably states that idiosyncratic income shocks can play only a modest role in matching the asset prices. As compared with 2.58% and 3.83% generated by GM (with elasticity of intertemporal substitution, EIS, ψ) for the risk-free rate and equity premium, respectively, ours generates without EIS (with the same discount rate and risk aversion, and LNIS $\delta = 5.0\%$) the risk-free rate of 2.20% and the equity premium of 3.27% (Table 5.2 (b)).

The comparison results between ours and CG (Table 5.2 (c)) and ours and Schmidt (Table 5.2 (d)) show the pivotal role of uninsurable idiosyncratic shocks in matching asset prices. CG incorporate uninsurable idiosyncratic income shocks to household consumption that are conditionally lognormal in the incomplete market where households are prevented from self insuring any part of their idiosyncratic income shocks.[40] Schmidt also focuses on

[40]CG models the idiosyncratic income shocks by a Poisson mixture of normals which becomes normal

idiosyncratic shocks to consumption growth that have state-dependent Gaussian and jump components (compound Poisson processes). Different from CG and Schmidt, our focus is on uninsurable and idiosyncratic shocks to income process (rather than consumption process) that are modeled by the LNIS.

Our risk-free rate and equity premium are both closer to the observed rates from 1929-2009 than CG without EIS (Table 5.2 (c)). This demonstrates that the consideration of jump risk such as the LNIS (rather than diffusive-type risk or Brownian-risk) can further improve the ability to match asset prices.

The ability of our asset pricing model is rather not enough to match asset prices from 1930-2008, while Schmidt dramatically lowers the risk-free rate and amplifies the equity premium, which are both very close to the historical rates (Table 5.2 (d)). Schmidt thoroughly investigates the asset pricing implications with state-dependent and idiosyncratic tail risk not only in the consumption process, but also in the labor income process, whereas we consider the LNIS only in the income process. Even though ours loses the ability to perfectly match asset prices unless we consider the uninsurable and idiosyncratic shocks in the consumption process as well, with the generally good and acceptable ability of our asset pricing model without resorting to EIS, it is much more analytically tractable than Schmidt's asset pricing framework.

6 Conclusion

We have developed an analytically tractable framework that generalizes Friedman's permanent income hypothesis (PIH) by considering the possibility of a large, negative income shock (LNIS). The generalized PIH with the optimal consumption and investment strategies helps attain consumption smoothing when the LNIS occurs. We have quantified precautionary savings and calculated the required amount of savings for consumption smoothing. We also have provided a general equilibrium analysis based on the Lucas-type pure exchange economy. The equilibrium interest rate with the LNIS is obtained in

conditional on the state variable.

closed-form. We find that the substantial amount of extra precautionary savings for consumption smoothing, driven by high-wealth people, should be reflected in the equilibrium asset pricing. We show that the equilibrium interest rate would therefore fall dramatically even when the chance of the LNIS is small; this result partly explains today's low interest rates. Finally, our equilibrium model's ability to match asset prices observed in the data is greatly improved with the LNIS.

Models	Model results		Required parameters				Number of parameters	MSE %
	Risk-free rate r	Equity premium $\mu - r$	LNIS δ	Recovery k	Discount rate β	Risk aversion γ		
Data	1.0%	6.0%	–	–	–	–	–	–
Lucas	9.3%	0.7%	–	–	5.0%	2.5	5	0.4849
Ours	3.8%	1.9%	5.0%	40%	5.0%	3	4	0.1233
Ours	1.0%	6.0%	5.0%	29%	1.0%	9.7	4	0

(a) Model results with required parameters (1889-1978)

Models	Model results		Required parameters					Number of parameters	MSE %
	Risk-free rate r	Equity premium $\mu - r$	LNIS δ	Recovery k	Discount rate β	Risk aversion γ	EIS ψ		
Data	1.58%	6.74%	–	–	–	–	–	–	–
GM	2.58%	3.83%	–	–	1.0%	5	0.6	6	0.0473
Ours	2.20%	3.27%	5.0%	40%	1.0%	5	–	4	0.0621
Ours	1.58%	6.74%	1.9%	40%	1.0%	10.31	–	4	0

(b) Model results with required parameters (1890-1997)

Models	Model results		Required parameters					Number of parameters	MSE
	Risk-free rate r	Equity premium $\mu - r$	LNIS δ	Recovery k	Discount rate β	Risk aversion γ	EIS ψ		
Data	0.6%	7.0%	–	–	–	–	–	–	–
CG	2.5%	4.9%	–	–	1.7%	5.05	1.10	14	0.0401
CG	4.2%	3.6%	–	–	1.3%	14.7	–	14	0.1226
Ours	3.1%	5.6%	1.0%	40%	1.3%	15	–	4	0.0411
Ours	1.5%	7.1%	1.0%	10%	1.3%	19	–	4	0.0041

(c) Model results with required parameters (1929-2009)

Models	Model results		Required parameters					Number of parameters	MSE
	Risk-free rate r	Equity premium $\mu - r$	LNIS δ	Recovery k	Discount rate β	Risk aversion γ	EIS ψ		
Data	0.57%	7.09%	–	–	–	–	–	–	–
Schmidt	0.46%	6.46%	8.0%	49%	2.55%	11	2	21	0.0020
Ours	2.90%	4.82%	5.0%	40%	2.55%	11	–	4	0.0529
Ours	1.02%	7.01%	5.0%	10%	2.55%	16	–	4	0.0001

(d) Model results with required parameters (1930-2008)

Table 3: Table compares our model with four representative general equilibrium models considering unhedgeable income risk by Lucas (Lucas, 1994), GM (Gomes and Michaelides, 2008), CG (Constantinides and Ghosh, 2017), and Schmidt (Schmidt, 2016).

References

Acemoglu, D., J. A. Robinson. 2015. The Rise and Decline of General Laws of Capitalism. *Journal of Economic Perspectives.* **29**, 3–28.

Ahn, S., K. J. Choi, and B. H. Lim. 2019. Optimal Consumption and Investment under Time-Varying Liquidity Constraints. *Journal of Financial and Quantitative Analysis.* **54**, 1643–1681.

Attanasio, O. P., M. Szekely. 2000. Household Saving in Developing Countries - Inequality, Demographics and All That: How Different are Latin America and South East Asia? Inter-American Development Bank Working Paper.

Ball, R., P. Brown. 1968. An Empirical Evaluation of Accounting Income Numbers. *Journal of Accounting Research.* **6**, 159–178.

Ball, R., G. Sadka, and R. Sadka. 2009. Aggregate Earnings and Asset Prices. *Journal of Accounting Research.* **47**, 1097–1133.

Bansan, R., A. Yaron. 2004. Risks for the Long-Run: A Potential Resolution of Asset Pricing Puzzles. *Journal of Finance.* **59**, 1481–1509.

Barro, R. 2006. Rare Disasters and Asset Markets in the Twentieth Century. *Quarterly Journal of Economics.* **121**, 823–866.

Basak, S. 1995. A General Equilibrium Model of Portfolio Insurance. *Review of Financial Studies.* **8**, 1059–1090.

Basak, S., D. Cuoco. 1998. An Equilibrium Model with Restricted Stock Market Participation. *Review of Financial Studies.* **11**, 309–341.

Basak, S., A. Shapiro. 2001. Value-at-Risk-Based Risk Management: Optimal Policies and Asset Prices. *Review of Financial Studies.* **14**, 371–405.

Bensoussan, A., B. G. Jang, and S. Park. 2016. Unemployment Risks and Optimal Retirement in an Incomplete Market. *Operations Research.* **64**, 1015–1032.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein. 2007. Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated. *Journal of Finance.* **62**, 2123–2167.

Bewley, T. 1977. The Permanent Income Hypothesis: A Theoretical Formulation. *Journal of Economic Theory.* **16**, 252–292.

Bodie, Z., R. C. Merton, and W. F. Samuelson. 1992. Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. *Journal of Economic Dynamics and Control.* **16**, 427–449.

Branger, N., M. Muck, F. T. Seifried, and S. Weisheit. 2017. Optimal Portfolios when Variances and Covariances can Jump. *Journal of Economic Dynamics and Control.* **85**, 59–89.

Buera, F. 2009. A Dynamic Model of Entrepreneurship with Borrowing Constraints. *Annals of Finance.* **5**, 443–464.

Calvet, L. E., P. Sodini. 2014. Twin Picks: Disentangling the Determinants of Risk-Taking in Household Portfolios. *Journal of Finance.* **69**, 867–906.

Campbell, J. Y. 1987. Does Saving Anticipate Declining Labor Income? An Alternative Test of the Permanent Income Hypothesis. *Econometrica.* **55**, 1249–1273.

Campbell, J. Y. 1999. Asset Prices, Consumption, and the Business Cycle. Handbook of Macroeconomics, vol. 1, chap. 19, North-Holland, Amsterdam.

Carroll, C. D., K. E. Dynan, and S. D. Krane. 2003. Unemployment Risk and Precautionary Wealth: Evidence from Households' Balance Sheets. *Review of Economics and Statistics.* **85**, 586–604.

Christensen, P. O., K. Larsen, and C. Munk. 2012. Equilibrium in Securities Markets with Heterogeneous Investors and Unspanned Income Risk. *Journal of Economic Theory.* **147**, 1035–1063.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout. 2005. Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies.* **18**, 491–533.

Constantinides, G. M., J. B. Donaldson, and R. Mehra. 2002. Junior Can't Borrow: A New Perspective on the Equity Premium Puzzle. *Quarterly Journal of Economics.* **117**, 269–296.

Constantinides, G. M., D. Duffie. 1996. Asset Pricing with Heterogeneous Consumers. *Journal of Political Economy.* **104**, 219-240.

Constantinides, G. M., A. Ghosh. 2017. Asset Pricing with Countercyclical Household Consumption Risk. *Journal of Finance.* **72**, 415–460.

Cox, J. C., C. Huang. 1989. Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory.* **49**, 33–83.

Deaton, A., C. Paxson. 2000. Growth and Saving among Individuals and Households. *Review of Economics and Statistics.* **82**, 212–225.

De Nardi, M., E. French, and J. B. Jones. 2010. Why Do the Elderly Save? The Role of Medical Expenses. *Journal of Political Economy.* **118**, 39–75.

De Nardi, M., G. Fella. Saving and Wealth Inequality. 2017. *Review of Economic Dynamics.* **26**, 280–300.

Duffie, D., W. Fleming, H. M. Soner, and T. Zariphopoulou. 1997. Hedging in Incomplete Markets with HARA utility. *Journal of Economic Dynamics and Control.* **21**, 753–782.

Durlauf, S. N., L. E. Blume. 2008. The New Palgrave Dictionary of Economics, 2nd edition.

Dumas, B., A. Lyasoff. 2012. Incomplete-Market Equilibria Solved Recursively on an Event Tree. *Journal of Finance.* **67**, 1897–1941.

Dybvig, P. H., H. Liu. 2010. Lifetime Consumption and Investment: Retirement and Constrained Borrowing. *Journal of Economic Theory.* **145**, 885–907.

Dynan, K., J. Skinner, and S. Zeldes. 2004. Do the Rich Save More? *Journal of Political Economy.* **112** 397–444.

Farhi, E., S. Panageas. 2007. Saving and Investing for Early Retirement: A Theoretical Analysis. *Journal of Financial Economics.* **83**, 87–121.

Frey, C. B., M. A. Osborne. 2017. The Future of Employment: How Susceptible are Jobs to Computerisation? *Technological Forecasting and Social Change.* **114**, 254–280.

Friedman, M. 1957. A Theory of the Consumption Function. Princeton University Press, Princeton.

Gabaix, X. 2012. Variable Rate Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance. *Quarterly Journal of Economics.* **127**, 645–700.

Garlappi, L., G. Skoulakis. 2010. Solving Consumption and Portfolio Choice Problems: The State Variable Decomposition Method. *Review of Financial Studies.* **23**, 3346–3400.

Gârleanu, N., S. Panageas. 2015. Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing. *Journal of Political Economy.* **123**, 670–685.

Gârleanu, N., S. Panageas. 2019. Heterogeneity and Asset Prices: A Different Approach. Working Paper.

Gomes, F., A. Michaelides. 2005. Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence. *Journal of Finance.* **60**, 869–904.

Gomes, F., A. Michaelides. 2008. Asset Pricing with Limited Risk Sharing and Heterogeneous Agents. *Review of Financial Studies.* **21**, 415–448.

Gomez, M. 2019. Asset Prices and Wealth Inequality. Working Paper.

Guvenen, F. 2009. A Parsimonious Macroeconomic Model For Asset Pricing. *Econometrica.* **77**, 1711–1750.

Guvenen, F., F. Karahan, S. Ozcan, and J. Song. 2015. What Do Data on Millions of U.S. Workers Reveal about Life-Cycle Earnings Risk?. University of Minnesota. Working Paper.

Heaton, J., D. J. Lucas. 1996. Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing. *Journal of Political Economy.* **104**, 443–487.

Jang, B. G., S. Park, and Y. Rhee. 2013. Optimal Retirement with Unemployment Risks. *Journal of Banking & Finance.* **37**, 3585–3604.

Jang, B. G., H. K. Koo, and S. Park. 2019. Optimal Consumption and Investment with Insurer Default Risk. *Insurance: Mathematics and Economics.* **88**, 44–56.

Jappelli, T. 1999. The Age-Wealth Profile and the Life-Cycle Hypothesis: A Cohort Analysis with a Time Series of Cross-Sections of Italian Households. *Review of Income and Wealth.* **45**, 57–75.

Jin, X., A. X. Zhang. 2012. Decomposition of Optimal Portfolio Weight in a Jump-Diffusion Model and Its Applications. *Review of Financial Studies.* **25**, 2877–2919.

Jin, X., D. Luo, and X. Zeng. 2017. Dynamic Asset Allocation with Uncertain Jump

Risks: A Pathwise Optimization Problem. *Mathematics of Operations Research.* **43**, 347–376.

Karatzas, I., J. P. Lehoczky, S. E. Shreve, and G.-L. Xu. 1991. Martingale and Duality Methods for Utility Maximization in an Incomplete Market. *SIAM Journal of Control and Optimization.* **29**, 702–730.

Kimball, M. S., M. D. Shapiro, T. Shumway, and J. Zhang. 2018. Portfolio Rebalancing in General Equilibrium. Working Paper.

Koo, H. K. 1998. Consumption and Portfolio Selection with Labor Income: A Continuous Time Approach. *Mathematical Finance.* **8**, 49–65.

Kou, S. G. 2002. A Jump-Diffusion Model for Option Pricing. *Management Science.* **48**, 1086–1101.

Krueger, D., H. Lustig. 2010. When is Market Incompleteness Irrelevant for the Price of Aggregate Risk (and When is It Not)? *Journal of Economic Theory.* **145**, 1–41.

Liu, J., F. A. Longstaff, and J. Pan. 2003. Dynamic Asset Allocation with Event Risk. *Journal of Finance.* **58**, 231–259.

Liu, J., F. A. J. Pan, and T. Wang. 2005. An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks. *Review of Financial Studies.* **18**, 131–164.

Low, H., C. Meghir, and L. Pistaferri. 2010. Wage Risk and Employment Risk over the Life Cycle. *American Economic Review.* **100**, 1432–1467.

Lucas, D. J. 1994. Asset Pricing with Undiversifiable Income Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle. *Journal of Monetary Economy.* **34**, 325–341.

Lucas, R. E. 1978. Asset Prices in an Exchange Economy. *Econometrica.* **46**, 1429–1445.

Lynch, A. W., S. Tan. 2011a. Labor Income Dynamics at Business-Cycle Frequencies: Implications for Portfolio Choice. *Journal of Financial Economics.* **101**, 333–359.

Lynch, A. W., S. Tan. 2011b. Explaining the Magnitude of Liquidity Premia: The Roles of Return Predictability, Wealth Shocks, and State-Dependent Transaction Costs. *Journal of Finance.* **66**, 1329–1368.

Maenhout, P. J. 2004. Robust Portfolio Rules and Asset Pricing. *Review of Financial*

Studies. **17**, 951–983.

Mehra, R., E. Prescott. 1985. The Equity Premium: A Puzzle. Journal of Monetary Economics. **15**, 145–161.

Merton, R. C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics.* **51**, 247–257.

Merton, R. C. 1971. Optimal Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory.* **3**, 373–413.

Milne, A. 2020. A Critical Covid 19 Economic Policy Tool: Retrospective Insurance. SSRN Working Paper.

Munk, C., C. Sørensen. 2010. Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics.* **96**, 433–462.

Piketty, T. 2014. Capital in the Twenty-First Century. Harvard University Press.

Polkovnichenko, V. 2007. Life-Cycle Portfolio Choice with Additive Habit Formation Preferences and Uninsurable Labor Income Risk. *Review of Financial Studies.* **20**, 83–124.

Ramsey, F. P. 1928. A Mathematical Theory of Saving. *Economic Journal.* **46**, 1429–1445.

Rietz, T. 1988. The Equity Risk Premium: A Solution. *Journal of Monetary Economics.* **22**, 117–131.

Ross, S. A. 1978. A Simple Approach to the Valuation of Risky Streams. *Journal of Business.* **51**, 453–475.

Schmidt, L. D. W. 2016. Climbing and Falling Off the Ladder: Asset Pricing Implications of Labor Market Event Risk. Working Paper.

Stokey, N. L., R. E. Lucas. 1989. Recursive Methods in Economic Dynamics. Harvard University Press, Cambridge, MA.

Wachter, J. A., M. Yogo. 2010. Why Do Household Portfolio Shares Rise in Wealth? *Review of Financial Studies.* **23**, 3929–3965.

Wang, N. 2003. Caballero Meets Bewley: The Permanent-Income Hypothesis in General Equilibrium. *American Economic Review.* **93**, 927–936.

Wang, C., N. Wang, and J. Yang. 2016. Optimal Consumption and Savings with Stochastic Income and Recursive Utility. *Journal of Economic Theory.* **165**, 292–331.

Weil, P. 1990. The Equity Premium Puzzle and the Risk-Free Rate Puzzle. *Journal of Monetary Economics.* **24**, 401–421.

Weil, P. 1992. Equilibrium Asset Prices with Undiversifiable Labor Income Risk. *Journal of Economic Dynamics and Control.* **16**, 769–790.

Online Appendix for "Negative Income Shocks and Asset Pricing"

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1 A Two-period Model

We start with a simple two-period model with a jump-type income shock. We consider a representative economic agent who aims to attain her optimal consumption and investment strategies over the two periods: period 0 and period 1. The agent dies at the end of period 1 and the probability of her survival until period 1 is δ_1. The objective of the agent is to maximize the following utility function by optimally controlling consumptions c_0 and c_1 at period 0 and at period 1, respectively:

$$v(c_0) + \delta_2 E[v(c_1)],$$

where v is a strictly increasing, strictly concave real-valued function defined on the set of positive real numbers, $0 < \delta_2 < 1$ is the subjective discount factor, and E denotes the expectation taken at period 0.

There are two tradable financial assets: a riskless bond and a risky stock. The riskless bond pays 1 at period 1 and its price is $\frac{1}{R}$ at period 0, where $R > 0$ is the risk-free interest rate. The price of a share of the risky stock is 1 at period 0 and can be u and d ($u > R > d > 0$) with probabilities π_u and $\pi_d = 1 - \pi_u$, respectively, at period 1. The agent obtains aggregate earnings at the rate of ϵ in each period. There is a jump shock in her earnings that would cause a significant downward jump in earnings from ϵ to 0 at period 1 with the probability of p. The probability distributions of the agent's mortality, the stock price, and the jump shock are assumed to be independent.

The budget constraint during period 1 is described as the following: for $i \in \{u, d\}$,

$$W_{1i} = \begin{cases} Rw_0^B + iw_0^S + \epsilon, & \text{if the income shock does not occur,} \\ Rw_0^B + iw_0^S, & \text{if the income shock occurs,} \end{cases}$$

where w_0^B is the dollar amount of savings invested in the riskless bond during period 0, and w_0^S is the dollar amount of savings invested in the risky stock during period 0.

The optimal consumption strategy c_{1i} at period 1 for $i \in \{u, d\}$ is to consume all of wealth W_{1i} available at period 1 i.e., $c_{1i} = W_{1i}$. The budget constraint during period 0 is given by

$$W_0 + \epsilon = c_0 + w_0^B + w_0^S,$$

where c_0 is the optimal consumption strategy at period 0.

The agent's optimization problem at period 0 is formulated by the following value function:

$$\begin{aligned}
&\max_{(w_0^B, w_0^S)} \Big[v\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) + \delta_2 E v\Big(W_1\Big)\Big] \\
&= \max_{(w_0^B, w_0^S)} \Big[v\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&\qquad + \delta(1-p)\Big\{\pi_u v\Big(Rw_0^B + uw_0^S + \epsilon\Big) + \pi_d v\Big(Rw_0^B + dw_0^S + \epsilon\Big)\Big\} \\
&\qquad + \delta p\Big\{\pi_u v\Big(Rw_0^B + uw_0^S\Big) + \pi_d v\Big(Rw_0^B + dw_0^S\Big)\Big\}\Big],
\end{aligned}$$

where $\delta \equiv \delta_1 \delta_2$. The first-order conditions for w_0^B and w_0^S are given by

$$\begin{aligned}
&v'\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&= (1-p)\delta R\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S + \epsilon\Big) + \pi_d v'\Big(Rw_0^B + dw_0^S + \epsilon\Big)\Big\} \\
&+ p\delta R\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S\Big) + \pi_d v'\Big(Rw_0^B + dw_0^S\Big)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&v'\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&= (1-p)\delta\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S + \epsilon\Big)u + \pi_d v'\Big(Rw_0^B + dw_0^S + \epsilon\Big)d\Big\} \\
&+ p\delta\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S\Big)u + \pi_d v'\Big(Rw_0^B + dw_0^S\Big)d\Big\},
\end{aligned}$$

respectively.

To obtain analytically tractable optimal strategies, we assume the simplest possible utility function: v is quadratic and it is given by

$$v(c) = c - \frac{\gamma}{2}c^2,$$

where γ is a positive constant. Then the first-order conditions become linear equations and we can derive in closed-form the optimal strategies as the following:

$$
\begin{aligned}
w_0^B = \Big[&\delta\{-1+\gamma(W_0+\epsilon)\}\{u\pi_u(u-R)-d\pi_d(R-d)\} \\
&+\delta\{(1-p)\gamma\epsilon-1\}\{(u\pi_u+d\pi_d-R)-\delta R\pi_u\pi_d(u-d)^2\}\Big] \\
&\Big/\gamma\Big[(1+\delta R^2)\{1+\delta(u^2\pi_u+d^2\pi_d)\}-\{1+\delta R(u\pi_u+d\pi_d)\}^2\Big],
\end{aligned}
$$

$$
\begin{aligned}
w_0^S = \Big[&\delta(u\pi_u+d\pi_d-R)\Big(R\{-1+\gamma(W_0+\epsilon)\}+\{(1-p)\gamma\epsilon-1\}\Big)\Big] \\
&\Big/\gamma\Big[\{1+\delta R(u\pi_u+d\pi_d)\}^2-\{1+\delta R^2(\pi_u+\pi_d)\}\{1+\delta(u^2\pi_u+d^2\pi_d)\}\Big].
\end{aligned}
$$

Notice that the premium term $u\pi_u + d\pi_d - R$ on the risky stock can be reasonably assumed to be positive. Then,

$$\frac{w_0^S}{\partial p} < 0.$$

This arguably states that the income shock reduces the dollar amount of savings invested in the risky stock.

However, we need to investigate whether this result does change relying on the assumptions under which the utility function is quadratic and only two periods rather than multi periods are considered. If we relax those assumptions by considering the well-known utility functions (the constant absolute or relative risk aversion utility function) or multi-period settings, to our best knowledge, the first-order conditions obtained when deriving optimal strategies turn out to be highly non-linear, so it is a considerable challenge to solve the problem analytically or even numerically.

Instead of the discrete time two-period model with the quadratic utility function, we will now develop a tractable continuous-time model with the constant relative risk aversion utility function, where all the optimal strategies and general equilibrium quantities are analytically tractable and derived in closed-form.

2 A Continuous-Time Model

2.1 The Building Blocks

The Benchmark Problem. In the context of the classic consumption utility maximization framework, the representative agent's consumption-savings model can be formulated by

$$V^B(w) \equiv \sup_c E\Big[\int_0^\infty e^{-rt} U\big(c(t)\big) dt\Big], \tag{1}$$

subject to the following wealth process of the agent:

$$dW(t) = \{rW(t) - c(t) + I\}dt, \ \ W(0) = w > -I/r,$$

and

$$W(t) > -\frac{I}{r}, \ \text{ for all } t \geq 0,$$

where r is the risk-free interest rate and $I > 0$ stands for reduced-form aggregate earnings.

Throughout the paper, we consider the following constant relative risk aversion (CRRA) utility function:

$$U\big(c(t)\big) = \frac{c(t)^{1-\gamma}}{1-\gamma},$$

where $\gamma > 0$ is the constant coefficient for an agent's relative risk aversion. We derive in closed-form the solution of the optimization problem of (1):

$$V^B(w) = \frac{1}{r}\frac{(rw+I)^{1-\gamma}}{1-\gamma}, \ \ w > -I/r.$$

Accordingly, we can obtain in closed-form the optimal consumption-savings strategy:

$$c(t) = rw + I. \tag{2}$$

The Permanent Income Hypothesis. We revisit the classic permanent income hypothesis (PIH) of Friedman (1957) and Modigliani and Brumberg (1954) based on the derived optimal consumption-savings strategy given in (2). The ratio of consumption to total available financial resources is constant:

$$\frac{c(t)}{w + I/r} = r,$$

where the denominator $w + I/r$ denotes total wealth (financial wealth w plus human capital I/r).[1] The marginal consumption with respect to financial wealth w is, thus, always constant, implying consumption smoothing. Intuitively, human capital strongly holds up total available financial resources and hence, people are able to attain the constant standard of living through consumption smoothing as the PIH predicts.[2]

The Optimal Consumption and Investment Problem in an Incomplete Market. We recollect the representative agent's optimal consumption and investment problem with a large, negative income shock (LNIS) stated in the manuscript entitled "Negative Income Shocks and Asset Pricing":

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{3}$$

where

$$K = A^{-\gamma},\ A = \frac{\gamma - 1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma},\ \beta_1 = r - \mu^I + (\sigma^I)^\top\theta,\ \theta = (\sigma^\top)^{-1}(\mu - r\mathbf{1}).$$

We consider an infinite-horizon economy with a single consumption good (the numeraire). Each representative agent has wealth $W(t)$ and invests $\pi(t)$ in the stock market, and saves her remaining wealth $W(t) - \pi(t)$ in the bond market. The agent also consumes $c(t)$ and receives $\xi I(t)$. The agent's dynamic wealth (budget) constraint is then: $W(0) = w > -\xi I/\beta_1$,

$$\begin{aligned} dW(t) &= d\pi(t) + d\{W(t) - \pi(t)^\top\mathbf{1}\} - c(t)dt + \xi I(t)dt \\ &= \pi(t)^\top\mu dt + \pi(t)^\top\sigma^\top dZ(t) + rW(t)dt - \pi(t)^\top r\mathbf{1}dt - c(t)dt + \xi I(t)dt \\ &= \{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top\sigma^\top dZ(t), \end{aligned} \tag{4}$$

where $\mathbf{1}$ is a vector of one's with dimensionality equal to the number of stocks, $\pi(t)$ is the dollar amount vector invested in each risky stock, and the borrowing limit is imposed by a wealth constraint

[1] The human capital presented here by I/r is the present value of future aggregate earnings discounted at the risk-free rate (Friedman, 1957; Hall, 1978).

[2] In reality, people can choose to finance very large expenditures such as buying a car, buying a house, sending their children to college, and so on, rather than directly spending cash. Here, human capital plays a key role in smoothing consumption as people can shift their earning power of labor from high-income periods to low-income periods of life. In high-income periods people can finance consumption needs by borrowing against their human capital, keeping cash in banks. In low-income periods people can use their savings for the most satisfying standard of living.

as follows. Similar to the wealth constraint with free borrowing against wages, the agent is allowed to borrow against the present value of her future wages (or the human capital):

$$W(t) > -\frac{\xi I(t)}{\beta_1}, \text{ for all } 0 \leq t < \tau, \tag{5}$$

where

$$\beta_1 = r - \mu^I + (\sigma^I)^\top \theta,$$

and τ is the random arrival of the LNIS driven by a Poisson jump process.

The technical details behind the derivation of the optimization problem (3) are given as follows. We have assumed in the manuscript that the LNIS is driven by a Poisson shock. On account of such an assumption, the problem is the same as the traditional Merton's (1969, 1971) case after the arrival of the Poisson shock. That is, the agent's problem after the occurrence of the LNIS is given by

$$V^A(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt\Big], \tag{6}$$

subject to

$$dW(t) = \{rW(t) - c(t) + k\xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top \sigma^\top dZ(t), \ W(0) = w > -\xi I/\beta_1, \tag{7}$$

$$W(t) > -\frac{\xi I(t)}{\beta_1}, \text{ for all } t \geq 0.$$

Note that the agent undergone the significant reduction in her aggregate earnings from $\xi I(t)$ to $k\xi I(t)$, $k \in (0, 1)$, as identified in (7). Ultimately, the problem (6) belongs to the conventional consumption utility-maximizing framework which aims to maximize the consumption of goods and services over the life cycle. Following Merton (1969, 1971), the problem is solved in closed-form:

$$V^A(w, I) = K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}. \tag{8}$$

Before integrating out the Poisson intensity δ, the original problem should be given by

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt + e^{-\beta\tau} V^A(w, I)\Big],$$

where τ represents the arrival of the Poisson shock. After integrating out the Poisson intensity δ, the problem stated above becomes the problem (3). The derivation details are as follows.

$$
\begin{aligned}
V(w,I) &= \sup_{(c,\pi)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(W(\tau),I(\tau))\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}K\frac{\{W(\tau)+k\xi I(\tau)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} \delta e^{-\delta s}\int_0^{s} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dtds \\
&\qquad + \int_0^{\infty} \delta e^{-\delta t}e^{-\beta t}K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}dt\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}\int_t^{\infty} \delta e^{-\delta s}dsdt \\
&\qquad + \int_0^{\infty} e^{-(\beta+\delta)}\delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}dt\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-(\beta+\delta)t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + \int_0^{\infty} e^{-(\beta+\delta)}\delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}dt\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big],
\end{aligned}
$$

2.2 The Solution

Problem Reformulation. Following Karatzas *et al.* (1991), we come up with a fictitious asset, $S^f(t)$, and its price dynamics are given by

$$dS^f(t) = \mu^f S^f(t)dt + \sigma^f S^f(t)dM(t),$$

where μ^f and σ^f are to be determined under the minimal local martingale measure and $M(t)$ is a compensated martingale process related to a Poisson shock. Specifically, the dynamics of $M(t)$ follow

$$dM(t) = -\delta dt + dN(t),$$

where $N(t)$ represents the Poisson shock. By defining the market price of the income shock (driven by the Poisson) shock as

$$\hat{\delta} \equiv \delta + \frac{\mu^f - r}{\sigma^f},$$

which will be called the income-shock-adjusted intensity. The state price densities are then characterized as

$$\xi^{\hat{\delta}}(t) \equiv \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big)\mathbf{1}_{\{\tau \leq t\}} - (\hat{\delta}-\delta)t \Big\} H(t), \tag{9}$$

where **1** is an indicator function that gives 1 if the Poisson shock occurs at time t and 0 otherwise, $H(t)$ is the standard state price density in complete markets under no arbitrage and its dynamics are given by

$$dH(t) = -H(t)\{rdt + \theta^{\top} dZ(t)\}, \;\; H(0) = 1.$$

We provide a lemma to convert the dynamic wealth constraint in (4) into the static wealth constraint as follows.

Lemma 2.1. *The dynamic wealth constraint in (4) can be converted into the following static wealth constraint:*

$$E\Big[\int_0^{\infty} e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big) dt\Big] \leq w. \tag{10}$$

Proof. See 3. **Q.E.D.**

With the help of Lemma 3.1, we are able to convert the original stochastic optimization problem (3) into the following static optimization problem:

$$V(w,I) = \sup_{(c,W)} E\Big[\int_0^{\infty} e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{11}$$

subject to

$$E\Big[\int_0^{\infty} e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big) dt\Big] \leq w.$$

We introduce another lemma to reformulate the problem (11) one more.

Lemma 2.2. *The static optimization problem of (11) can be reformulated as*

$$\begin{aligned} V(w,I) &= \inf_{(\lambda,\hat{\delta})} \{J^{\hat{\delta}}(\lambda, I) + \lambda w\} \\ &= \inf_{\lambda}\{\inf_{\hat{\delta}} J^{\hat{\delta}}(\lambda, I) + \lambda w\} \\ &\equiv \inf_{\lambda}\{J(\lambda, I) + \lambda w\}, \end{aligned} \tag{12}$$

where the indirect value function $J^{\hat{\delta}}(\lambda, I)$ is given by

$$J^{\hat{\delta}}(\lambda, I) = (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^{\infty} e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + \Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big]$$
$$\equiv (\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z) \tag{13}$$

with $z = \lambda(\xi I)^{\gamma}$, where $\tilde{E}$ is the expectation under the new probability measure defined as

$$\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega) + (1-\gamma)(\sigma^I)^{\top}Z(t,\omega)\Big)dP(\omega) \ \textit{for all} \ \ A \in \mathcal{F}$$

with the new Brownian motion process $\tilde{Z}$ given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I dt + Z(t),$$

$\Gamma^{\hat{\delta}}(t)$ *is a new state variable defined by*

$$\Gamma^{\hat{\delta}}(t) \equiv \lambda e^{(\beta+\delta-\hat{\delta})t}H(t)(\xi I(t))^{\gamma},$$

and its dynamics are given by

$$d\Gamma^{\hat{\delta}}(t) = \Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\},$$

$$\beta_1^{\hat{\delta}} = r - \mu^I + (\sigma^I)^{\top}\theta + \hat{\delta} - \delta,$$
$$\beta_2 = \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$
$$\beta_3 = \gamma(\sigma^I)^{\top} - \theta^{\top}.$$

Proof. See 3. **Q.E.D.**

Minimal Local Martingale Measure. In order to solve the reformulated problem (12), the income-shock-adjusted intensity $\hat{\delta}$ should be determined appropriately to find the minimal local martingale measure, guaranteeing the uniqueness of the state price density. Using the dynamic programming approach of Bensoussan *et al.* (2016), we can determine $\hat{\delta}$ uniquely and explicitly.

Lemma 2.3. *The income-shock-adjusted intensity $\hat{\delta}$ is determined uniquely by*

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma}\frac{\delta K}{z}, \tag{14}$$

where $G(z) = -\varphi'_\delta(z) + 1/\beta_1$ solves to the following non-linear differential equation:

$$\begin{aligned} &-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ &\qquad + \beta_1 G(z) + \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}. \end{aligned} \tag{15}$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \;\; \textit{and} \;\; G'(\overline{z}) = 0.$$

Proof. See 3. **Q.E.D.**

Now, it remains to solve the non-linear differential equation (15) to characterize the income-shock-adjusted intensity $\hat{\delta}$ explicitly.

Proposition 2.1. *A general solution of the differential equation in (15) is given by*

$$\begin{aligned} G(z) =& \frac{1}{\hat{A} + \delta} z^{-1/\gamma} + B^*_\delta z^{-\alpha^*_\delta} \\ &+ \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha^*_\delta)(1-\gamma)} \Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ &+ (\alpha^*_\delta - 1) z^{-\alpha^*_\delta} \int_z^{\overline{z}} \mu^{\alpha^*_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big], \end{aligned} \tag{16}$$

where $\hat{A}$ is given by

$$\hat{A} = \frac{\gamma - 1}{\gamma}\Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},$$

B^*_δ and $\overline{z}$ are the two constants to be determined by the boundary conditions:

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \;\; \textit{and} \;\; G'(\overline{z}) = 0,$$

and $\alpha_\delta > 1$ *and* $-1 < \alpha_\delta^* < 0$ *are the two roots to the following characteristic equation:*

$$F(\alpha;\delta) \equiv -\frac{1}{2}||\beta_3||^2\alpha(\alpha-1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Proof. See 3. **Q.E.D.**

3 Technical Details of Solution

In the main manuscript, we have provided analytically tractable results for the optimal strategies. In this subsection, the associated technical details are provided. To derive the main results, the most difficult part is to solve the stochastic optimization problem (3). More specifically, the problem is inclined to involve unwanted unlimited downside utilities with some possibilities. This is because the term in (3) that involves the Poisson intensity δ is highly likely to be caught up in $-\infty$ when borrowing is allowed against human capital, i.e., as wealth $W(t)$ approaches $-\xi I(t)/\beta_1$. The simplest approach to remedy this problem is to introduce a time-varying lower bound of wealth:

$$W(t) > -L(t) > -\frac{k\xi I(t)}{\beta_1}, \ \text{ for all } t \geq 0, \tag{17}$$

where $L(t)$ is a given nonnegative time-varying function, but not completely arbitrary. With the help of the lower bound of wealth in (17), the problem (3) is now well defined.

The State Price Density. Solving the stochastic optimization problem (3) involves many steps. One of the most important steps is to explicitly characterize the unique state price density in an incomplete market. This is a daunting task, as there exist infinitely many possible candidates for the state price density in our economic setting, where not only the market risk, but also the income shock give rise to market incompleteness. In order to address the issue associated with market incompleteness, we rely on the fictitious completion of Karatzas *et al.* (1991). Specifically, we come up with a fictitious asset, $S^f(t)$, and its price dynamics are given by

$$dS^f(t) = \mu^f S^f(t)dt + \sigma^f S^f(t)dM(t),$$

where μ^f and σ^f are to be determined with the minimal local martingale measure and $M(t)$ is a compensated martingale process related to a Poisson shock. Specifically, the dynamics of $M(t)$ follow

$$dM(t) = -\delta dt + dN(t),$$

where $N(t)$ represents the Poisson shock. Given the following income-shock-adjusted intensity

$$\hat{\delta} \equiv \delta - \frac{\mu^f - r}{\sigma^f},$$

the state price densities are then characterized as

$$\xi^{\hat{\delta}}(t) \equiv \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big)\mathbf{1}_{\{\tau \le t\}} - (\hat{\delta} - \delta)t \Big\} H(t),$$

where **1** is an indicator function that gives 1 if the Poisson shock occurs at time t and 0 otherwise, $H(t)$ is the standard state price density in complete markets under no arbitrage and its dynamics are given by

$$dH(t) = -H(t)\{rdt + \theta^\top dZ(t)\}, \;\; H(0) = 1.$$

Having characterized the state price densities in the presence of the income shock, we introduce one important lemma to convert the dynamic wealth constraint in (4) into the static wealth constraint as follows.

Lemma 3.1. *The dynamic wealth constraint in (4) can be converted into the following static wealth constraint:*

$$E\Big[\int_0^\infty e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big) dt\Big] \le w. \tag{18}$$

Proof. By applying Itô's formula to $d\big(e^{-rt}W(t)\big)$ yields

$$d\big(e^{-rt}W(t)\big) = -e^{-rt}\{c(t) - \xi I(t)\}dt + e^{-rt}\pi(t)^\top d\tilde{Z}(t), \tag{19}$$

where $\tilde{Z}$ is the Brownian motion process under the new martingale measure with respect to the state price density $\xi^{\hat{\delta}}(t)$. By Girsanov's theorem, the new probability measure is defined by

$$\tilde{P}(A) \equiv \int_A e^{rt}\xi^{\hat{\delta}}(t,\omega)dP(\omega) \;\; \text{for all} \;\; A \in \mathcal{F} \tag{20}$$

and the Brownian motion process $\tilde{Z}$ follows

$$\tilde{Z}(t) \equiv \theta dt + Z(t).$$

Integrating both sides of (19) from 0 to τ,

$$\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau) = w + \int_0^\tau e^{-rt}\pi(t)^\top d\tilde{Z}(t).$$

Taking expectation $\tilde{E}$ under the new martingale measure,

$$\tilde{E}\Big[\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau)\Big] \leq w.$$

Changing the martingale measure into the physical measure using the relationship of (20),

$$E\Big[\int_0^\tau \xi^{\hat{\delta}}(t)\Big(c(t) - \xi I(t)\Big)dt + \xi^{\hat{\delta}}(\tau)W(\tau)\Big] \leq w.$$

Integrating out the Poisson intensity δ with respect to τ using the conditional expectation completes the proof of the lemma. **Q.E.D.**

The Static Optimization Problem. With the help of Lemma 3.1, we solve the stochastic optimization problem (3) by using the martingale representation approach in an incomplete market (Karatzas *et al.*, 1991), thereby converting the problem into the following static optimization problem:

$$V(w, I) = \sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \qquad (21)$$

subject to

$$E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c(t) - \xi I(t) + \hat{\delta}W(t)\Big)dt\Big] \leq w.$$

The optimization problem given by (21) seems to be almost impossible to be solved analytically or even numerically. We introduce a lemma to reformulate the optimization problem as follows.

Lemma 3.2. *The static optimization problem of (21) is reformulated as*

$$\begin{aligned} V(w,I) &= \inf_{(\lambda,\hat{\delta})}\{J^{\hat{\delta}}(\lambda,I)+\lambda w\} \\ &= \inf_{\lambda}\{\inf_{\hat{\delta}} J^{\hat{\delta}}(\lambda,I)+\lambda w\} \\ &\equiv \inf_{\lambda}\{J(\lambda,I)+\lambda w\}, \end{aligned} \tag{22}$$

where the indirect value function $J^{\hat{\delta}}(\lambda, I)$ is given by

$$\begin{aligned} J^{\hat{\delta}}(\lambda,I) &= (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}+\Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big] \\ &\equiv (\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z) \end{aligned} \tag{23}$$

with $z=\lambda(\xi I)^{\gamma}$, where $\tilde{E}$ is the expectation under the new probability measure defined by

$$\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega)+(1-\gamma)(\sigma^I)^\top dZ(t,\omega)\Big)dP(\omega) \ \text{ for all } \ A\in\mathcal{F}$$

with the new Brownian motion process $\tilde{Z}$ given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I dt + Z(t),$$

$\Gamma^{\hat{\delta}}(t)$ is a new state variable defined by

$$\Gamma^{\hat{\delta}}(t) \equiv \lambda e^{(\beta+\delta-\hat{\delta})t}H(t)(\xi I(t))^{\gamma},$$

and its dynamics are given by

$$d\Gamma^{\hat{\delta}}(t) = \Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}}-\beta_2)dt+\beta_3 d\tilde{Z}(t)\},$$

$$\begin{aligned} \beta_1^{\hat{\delta}} &= r-\mu^I+(\sigma^I)^\top\theta+\hat{\delta}-\delta, \\ \beta_2 &= \beta-\mu^I(1-\gamma)+\frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2, \\ \beta_3 &= \gamma(\sigma^I)^\top-\theta^\top. \end{aligned}$$

Proof. Using the standard Lagrangian approach, we can construct the indirect value function, $J^{\hat{\delta}}(\lambda, I)$, and it is given by

$$
\begin{aligned}
J^{\hat{\delta}}(\lambda, I) \equiv \sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\
-\lambda E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c(t) - \xi I(t) + \hat{\delta}W(t)\Big)dt\Big].
\end{aligned} \tag{24}
$$

Applying the first-order conditions for consumption $c(t)$ and wealth $W(t)$ gives rise to

$$
\begin{aligned}
c(t) &= \Big(\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}, \\
W(t) &= \Big(\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}}{\delta}\Big)^{-1/\gamma}K^{1/\gamma} - k\xi I(t)/\beta_1.
\end{aligned} \tag{25}
$$

The indirect value function in (24) can be rewritten when the above first-order conditions for consumption and wealth are substituted in:

$$
\begin{aligned}
J^{\hat{\delta}}(\lambda, I) = E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{1-1/\gamma} \\
+ \frac{\gamma}{1-\gamma}(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{1-1/\gamma} \\
+ \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)\xi I(t)\Big)\Big\}dt\Big].
\end{aligned} \tag{26}
$$

We introduce a new state variable to reformulate the indirect value function in (26). Specifically,

$$
\Gamma^{\hat{\delta}}(t) \equiv \lambda e^{(\beta+\delta-\hat{\delta})t}H(t)(\xi I(t))^{\gamma}.
$$

The indirect value function in (24) can be reformulated as the function of $\Gamma^{\hat{\delta}}(t)$:

$$
\begin{aligned}
J^{\hat{\delta}}(\lambda, I) = E\Big[\int_0^\infty (\xi I(t))^{1-\gamma}e^{-(\beta+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + (\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}\Big) \\
+ \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big].
\end{aligned}
$$

By Girsanov's Theorem, the new probability measure can be defined by

$$
\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega) + (1-\gamma)(\sigma^I)^\top dZ(t,\omega)\Big)dP(\omega) \text{ for all } A \in \mathcal{F}
$$

and the new Brownian motion process $\tilde{Z}$ is given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I dt + Z(t).$$

The dynamics of the new state variable follow

$$d\Gamma^{\hat{\delta}}(t) = \Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\},$$

where

$$\begin{aligned}
\beta_1^{\hat{\delta}} &= r - \mu^I + (\sigma^I)^\top \theta + \hat{\delta} - \delta, \\
\beta_2 &= \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2, \\
\beta_3 &= \gamma(\sigma^I)^\top - \theta^\top.
\end{aligned}$$

As a result, the indirect value function is given by

$$\begin{aligned}
J^{\hat{\delta}}(\lambda, I) &= (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1 + (\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big] \\
&\equiv (\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z),
\end{aligned}$$

where $z = \lambda(\xi I)^\gamma$. Following Karatzas *et al.* (1999), the stochastic optimization problem (3) or equivalently, the static optimization problem (21) essentially derives from the indirect value function in (24) by the following relationship:

$$\begin{aligned}
V(w, I) &= \inf_{(\lambda,\hat{\delta})}\{J^{\hat{\delta}}(\lambda, I) + \lambda w\} \\
&= \inf_\lambda\{\inf_{\hat{\delta}} J^{\hat{\delta}}(\lambda, I) + \lambda w\} \\
&\equiv \inf_\lambda\{J(\lambda, I) + \lambda w\},
\end{aligned}$$

which completes the proof. **Q.E.D.**

The Income-Shock-Adjusted Intensity. In the lemma, $\hat{\delta}$ is to be determined to find out the minimal local martingale measure, guaranteeing the uniqueness of the state price density through which the risk neutral probability measure can be constructed to give more weight to unwanted events resulting from the LNIS relative to the physical (or the original) probability measure. The following lemma determines the unique $\hat{\delta}$.

Lemma 3.3. *The income-shock-adjusted intensity $\hat{\delta}$ is determined uniquely by*

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z}, \tag{27}$$

where $G(z) = -\varphi'_{\hat{\delta}}(z) + 1/\beta_1$ solves to the following non-linear differential equation:

$$-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \ \ 0 < z < \overline{z}. \tag{28}$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \textit{and} \ G'(\overline{z}) = 0.$$

Proof. The function $\varphi_{\hat{\delta}}(z)$ in (23) should satisfy by Feynman-Kac's formula the following non-linear ordinary differential equation:

$$\inf_{\hat{\delta}} \Big[\frac{1}{2}||\beta_3||^2 z^2 \varphi''_{\hat{\delta}}(z) - (\beta_1^{\hat{\delta}} - \beta_2) z \varphi'_{\hat{\delta}}(z) - (\beta_2 + \delta)\varphi_{\hat{\delta}}(z) \\ + \frac{\gamma}{1-\gamma}\Big(1 + (\delta K)^{1/\gamma} \hat{\delta}^{1-1/\gamma}\Big) z^{1-1/\gamma} + \Big(1 + \frac{\hat{\delta} k}{\beta_1}\Big) z\Big] = 0, \ \ 0 < z < \overline{z}, \tag{29}$$

where

$$\beta_1^{\hat{\delta}} = r - \mu^I + (\sigma^I)^\top \theta + \hat{\delta} - \delta,$$
$$\beta_2 = \beta - \mu^I (1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$
$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

and $\overline{z}$ is to be determined according to the boundary conditions (or the value matching and smooth pasting conditions) given by

$$\varphi'_{\hat{\delta}}(\overline{z}) = \frac{L}{\xi I}, \ \ \varphi''_{\hat{\delta}}(\overline{z}) = 0.$$

Note that the technical details behind the boundary conditions stated above are essentially the same as Dybvig and Liu (2011). Applying the first-order condition for $\hat{\delta}$ leads to

$$\hat{\delta} = \Big(-\varphi'_{\hat{\delta}}(z) + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z}. \tag{30}$$

When the above first-order condition is substituted in (29), the differential equation is rewritten as

$$\frac{1}{2}||\beta_3||^2 z^2 \varphi''_{\hat{\delta}}(z) - (\beta_1 - \delta - \beta_2) z \varphi'_{\hat{\delta}}(z) - (\beta_2 + \delta)\varphi_{\hat{\delta}}(z) + \frac{\gamma}{1-\gamma} z^{1-1/\gamma} + z + \frac{\delta K}{1-\gamma}\Big(-\varphi'_{\hat{\delta}}(z) + \frac{k}{\beta_1}\Big)^{1-\gamma} = 0, \;\; 0 < z < \overline{z}. \tag{31}$$

From now on, we will carry out several transformations to simplify the differential equation given in (31). We denote $-\varphi'_{\hat{\delta}}(z)$ by $\tilde{G}(z)$. By differentiating the both sides of (31) with respect to z, the differential equation (31) is restated with $\tilde{G}(z)$ as follows:

$$-\frac{1}{2}||\beta_3||^2 z^2 \tilde{G}''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z \tilde{G}'(z) + \beta_1 \tilde{G}(z) + 1 + \delta K\Big(\tilde{G}(z) + \frac{k}{\beta_1}\Big)^{-\gamma} \tilde{G}'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}, \tag{32}$$

with the boundary conditions

$$\tilde{G}(\overline{z}) = -\frac{L}{\xi I} \text{ and } \tilde{G}'(\overline{z}) = 0.$$

We also denote $\tilde{G}(z) + 1/\beta_1$ by $G(z)$. Then the differential equation (32) is rewritten as

$$-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) + \beta_1 G(z) + \delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}.$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \text{ and } G'(\overline{z}) = 0.$$

Finally, $\hat{\delta}$ given in (30) is rewritten as a function of $G(z)$, which completes the proof. **Q.E.D.**

The Indirect Value Function. The relationship (22) shows that the optimization problem in (3) or equivalently, the problem (21) is solved by deriving the indirect value function in (23) together with the income-shock-adjusted intensity $\hat{\delta}$ in the lemma. Going forward, we devote our full attention to solving the non-linear differential equation given in (28) to derive the indirect value function $J(\lambda, I)$ given in (23).

Now, we introduce the important proposition to derive a general solution to the differential equation in (28).

Proposition 3.1. *A general solution to the differential equation in (28) is given by*

$$
\begin{aligned}
G(z) =& \frac{1}{\hat{A}+\delta} z^{-1/\gamma} + (B_\delta^*)^{-\alpha_\delta^*} \\
&+ \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \Big[(\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2} \Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \\
&+ (\alpha_\delta^* - 1) z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2} \Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \Big],
\end{aligned} \tag{33}
$$

where $\hat{A}$ is given by

$$\hat{A} = \frac{\gamma - 1}{\gamma} \Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma} \Big) + \frac{\beta_2}{\gamma},$$

B_δ^* and $\overline{z}$ are the two constants to be determined by the boundary conditions:

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \text{ and } \ G'(z) = 0,$$

and $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equations:

$$F(\alpha; \delta) \equiv -\frac{1}{2} ||\beta_3||^2 \alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Proof. We conjecture a general solution of the equation (28) as

$$G(z) = \frac{1}{\hat{A}+\delta} z^{-1/\gamma} + \eta(z) z^{-\alpha_\delta} + \eta^*(z) z^{-\alpha_\delta^*}, \tag{34}$$

subject to

$$\eta'(z) z^{-\alpha_\delta} + (\eta^*(z))' z^{-\alpha_\delta^*} = 0,$$

where $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equations:

$$F(\alpha; \delta) \equiv -\frac{1}{2} ||\beta_3||^2 \alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Direct calculations of the first and second derivative of G result in

$$G'(z) = -\frac{1}{\gamma(\hat{A}+\delta)} z^{-1/\gamma - 1} - \alpha_\delta \eta(z) z^{-\alpha_\delta - 1} - \alpha_\delta^* \eta^*(z) z^{-\alpha_\delta^* - 1}$$

and

$$G''(z) = \Big(1+\frac{1}{\gamma}\Big)\frac{1}{\gamma(\hat{A}+\delta)}z^{-1/\gamma-2} - \alpha_\delta\eta'(z)z^{-\alpha_\delta-1} + \alpha_\delta(\alpha_\delta+1)\eta(z)z^{-\alpha_\delta-2}$$
$$- \alpha_\delta^*(\eta^*(z))'z^{-\alpha_\delta^*-1} + \alpha_\delta^*(\alpha_\delta^*+1)\eta^*(z)z^{-\alpha_\delta^*-2}.$$

Using the general solution (34) and the derivatives of G stated above, the first three terms of left-hand side in (28) become

$$\begin{aligned}
&-\frac{1}{2}||\beta_3||^2z^2G''(z) - (||\beta_3||^2+\beta_2+\delta-\beta_1)zG'(z) + \beta_1 G(z)\\
&= z^{-1/\gamma} + \frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta}\eta'(z)\\
&= z^{-1/\gamma} - \frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta^*}(\eta^*(z)).
\end{aligned}$$

As a result, the differential equation (28) simplifies to the following: for $0<z<\overline{z}$,

$$\frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta}\eta'(z) = -\delta K\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(z)$$

and

$$\frac{||\beta_3||^2}{2}(\alpha_\delta-\alpha_\delta^*)z^{1-\alpha_\delta^*}(\eta^*(z)) = \delta K\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(z).$$

Integrating the both sides of the above two relationships from 0 to z and from z to $\overline{z}$ allows $\eta(z)$ and $\eta^*(z)$ to be expressed as an integral form:

$$\eta(z) = -\frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)}\int_0^z \mu^{\alpha_\delta-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu$$

and

$$\eta^*(z) = \eta^*(\overline{z}) - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu.$$

Therefore, the general solution (34) also can be expressed as an integral form:

$$G(z) = \frac{1}{\hat{A}+\delta}z^{-1/\gamma} + \eta^*(\overline{z})z^{-\alpha_\delta^*} - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)}\Big[z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu$$
$$+ z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-1}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)d\mu\Big]. \quad (35)$$

Note that

$$\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\mu)=\frac{d}{d\mu}\Big\{\frac{1}{1-\gamma}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}\Big\}.$$

Using the integration by parts, the general solution (35) can be restated as follows:

$$\begin{aligned}G(z)=&\frac{1}{\hat{A}+\delta}z^{-1/\gamma}+\Big\{\eta^*(\overline{z})+\overline{z}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{z})-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}\Big\}z^{-\alpha_\delta^*}\\&+\frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\\&+(\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}}\mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big].\end{aligned}\tag{36}$$

Defining a constant B_δ^* as

$$B_\delta^*\equiv\eta^*(\overline{z})+\overline{z}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{z})-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}.$$

Finally, we obtain the general solution in closed-form:

$$\begin{aligned}G(z)=&\frac{1}{\hat{A}+\delta}z^{-1/\gamma}+(B_\delta^*)^{-\alpha_\delta^*}\\&+\frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\\&+(\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}}\mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big],\end{aligned}$$

which completes the proof. **Q.E.D.**

Optimal Consumption and Investment Strategies. Now, we are ready to derive the analytic results of optimal consumption and investment strategies.

Theorem 3.1. *The optimal consumption strategy $c(t)$ and the optimal investment strategy $\pi(t)$ of the agent with the LNIS are derived in closed-form:*

$$c(t)=(\hat{A}+\delta)\Big(w+\frac{\xi I}{\beta_1}-\xi IB_\delta^*z^{-\alpha_\delta^*}-IP\Big),\tag{37}$$

$$
\begin{aligned}
\pi(t) = & \frac{1}{\gamma}\sigma^{-1}\theta w \\
& + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*} \\
& - \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma} \Big/ c(t)^{-\gamma} + (\gamma\alpha_\delta - 1) \times \mathit{IP1} + (\gamma\alpha_\delta^* - 1) \times \mathit{IP2}\Big],
\end{aligned} \tag{38}
$$

where

$$
\hat{A} = \frac{\gamma - 1}{\gamma}\Big(\beta_1 + \frac{||\beta_3||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},
$$

B_δ^* and $\overline{z}$ are the two constants to be determined by the boundary conditions

$$
G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \ \text{ and } \ G'(\overline{z}) = 0,
$$

$\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equation:

$$
F(\alpha;\delta) \equiv -\frac{1}{2}||\beta_3||^2\alpha(\alpha - 1) + \alpha(\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0,
$$

$G(z)$ satisfies the following non-linear differential equation: for $0 < z < \overline{z}$,

$$
\begin{aligned}
-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) & \\
+ \beta_1 G(z) + \delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) & = z^{-1/\gamma},
\end{aligned}
$$

with

$$
\begin{aligned}
\beta_2 &= \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2, \\
\beta_3 &= \gamma(\sigma^I)^\top - \theta^\top,
\end{aligned}
$$

IP represents the integral parts of LNIS-induced precautionary savings and it is given by

$$
IP = IP1 + IP2,
$$

$$
\begin{aligned}
IP1 &= \frac{2\delta K(\alpha_\delta - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu < 0, \\
IP2 &= \frac{2\delta K(\alpha_\delta^* - 1)\xi I}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu > 0,
\end{aligned} \tag{39}
$$

and the LNIS-induced precautionary savings (LNIS-PS) are given by

$$\begin{aligned}\textit{LNIS-PS} = & -\delta\Big(w + \frac{\xi I}{\beta_1}\Big) + \Big(\hat{A} + \delta - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha^*_\delta - 1)\Big)\xi I B^*_\delta z^{-\alpha^*_\delta} \\ & + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c(t)^{-\gamma} \\ & + \Big(\hat{A} + \delta - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_\delta - 1)\Big) \times \textit{IP1} \\ & + \Big(\hat{A} + \delta - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha^*_\delta - 1)\Big) \times \textit{IP2}.\end{aligned} \tag{40}$$

Proof. With the first-order condition (27) for $\hat{\delta}$, the first-order conditions for consumption $c(t)$ in (25) can be rewritten as

$$c(t) = \xi I(t)\Gamma^{\hat{\delta}}(t)^{-1/\gamma}, \tag{41}$$

where $\Gamma^{\hat{\delta}}(t)$ is given by

$$\Gamma^{\hat{\delta}(t)} = \lambda e^{(\beta+\delta-\hat{\delta})t} H(t)(\xi I(t))^\gamma,$$

and

$$\hat{\delta} = \Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{\Gamma^{\hat{\delta}}(t)}$$

By the principle of dynamic programming, it is convenient to express the consumption as a function of initial variable z:

$$c(t) = c(0) = \xi I z^{-1/\gamma}. \tag{42}$$

From the relationship (22) between the value function and the indirect value function, applying the first-order condition for λ results in

$$\begin{aligned} w &= -J_\lambda(\lambda, I) \\ &= -\xi I \varphi'_{\hat{\delta}}(z) \\ &= \xi I \tilde{G}(z) \\ &= \xi I\Big(G(z) - \frac{1}{\beta_1}\Big), \end{aligned} \tag{43}$$

accordingly,

$$G(z) = \frac{w}{\xi I} + \frac{1}{\beta_1}.$$

A little rearrangement of the general solution (33) leads to

$$
\begin{aligned}
z^{-1/\gamma} &= (\hat{A}+\delta)\Big[G(z) - B_\delta^* z^{-\alpha_\delta^*} \\
&\qquad - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu \\
&\qquad + (\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big] \\
&= (\hat{A}+\delta)\Big[\frac{w}{\xi I}+\frac{1}{\beta_1} - B_\delta^* z^{-\alpha_\delta^*} \\
&\qquad - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu \\
&\qquad + (\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big].
\end{aligned}
$$

Therefore, the first-order condition for consumption $c(t)$ in (41) allows the following optimal consumption strategy:

$$
c(t) = (\hat{A}+\delta)\Big(w + \frac{\xi I}{\beta_1} - \xi I B_\delta^* z^{-\alpha_\delta^*} - \text{IP}\Big],
$$

where IP represents the integral parts of LNIS-induced precautionary savings and it is given by

$$
\begin{aligned}
&\frac{2\delta K\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu \\
&\qquad\qquad + (\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big].
\end{aligned}
$$

For simplicity, IP is rewritten as

$$
\text{IP} = \text{IP1} + \text{IP2},
$$

where

$$
\text{IP1} = \frac{2\delta K(\alpha_\delta-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu < 0,
$$

$$
\text{IP2} = \frac{2\delta K(\alpha_\delta^*-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu > 0.
$$

It remains to derive the optimal investment strategy. A little rearrangement of the relationship in (43) gives

$$
\frac{w}{\xi I} = G(z) - \frac{1}{\beta_1},
$$

or equivalently,

$$\frac{W(t)}{\xi I(t)} = G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1}. \tag{44}$$

By applying It$\hat{o}$'s formula to the left hand side of the above relationship,

$$\begin{aligned}
d\Big(\frac{W(t)}{\xi I(t)}\Big) &= dW(t)\frac{1}{\xi I(t)} + W(t)d\Big(\frac{1}{\xi I(t)}\Big) + dW(t)d\Big(\frac{1}{\xi I(t)}\Big) \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + W(t)\Big[-(\xi I(t))^{-2}dI(t) + (\xi I(t))^{-3}(dI(t))^2\Big] \\
&\quad + \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big] \\
&\quad \times \Big[-(\xi I(t))^{-2}dI(t) + (\xi I(t))^{-3}(dI(t))^2\Big] \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + W(t)\Big[-(\xi I(t))^{-1}\{\mu^I dt + (\sigma^I)^\top dZ(t)\} + (\xi I(t))^{-1}||\sigma^I||^2 dt\Big] \\
&\quad + \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big] \\
&\quad \times \Big[-(\xi I(t))^{-1}\{\mu^I dt + (\sigma^I)^\top dZ(t)\} + (\xi I(t))^{-1}||\sigma^I||^2 dt\Big] \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + \frac{W(t)}{\xi I(t)}\Big[-(\mu^I - ||\sigma^I||^2)dt - (\sigma^I)^\top dZ(t)\Big] - (\xi I(t))^{-1}\pi(t)^\top\sigma^\top\sigma^I dt \\
&= \frac{1}{\xi I(t)}\Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1}) - W(t)(\mu^I - ||\sigma^I||^2) - \pi(t)^\top\sigma^\top\sigma^I\Big\}dt \\
&\qquad + \{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top\}dZ(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\Big\{\{r - \mu^I + ||\sigma^I||^2\}W(t) - c(t) + \xi I(t) + \pi(t)^\top\{\mu - r\mathbf{1} - \sigma^\top\sigma^I\}\Big\}dt \\
&\qquad + \{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top\}dZ(t)\Big].
\end{aligned} \tag{45}$$

By applying It$\hat{o}$'s formula to the right hand side of the relationship (44),

$$
\begin{aligned}
dG(\Gamma^{\hat{\delta}}(t)) &= G'(\Gamma^{\hat{\delta}}(t))d\Gamma^{\hat{\delta}}(t) + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(d\Gamma^{\hat{\delta}}(t))^2 \\
&= G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 dt \\
&= G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3\{-(1-\gamma)\sigma^I dt + dZ(t)\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 dt \\
&= \Big\{ - G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{(\beta_1^{\hat{\delta}} - \beta_2) + (1-\gamma)\beta_3\sigma^I\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2\Big\}dt \\
&\quad + G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\beta_3 dZ(t).
\end{aligned}
\tag{46}
$$

Equating each term of $dZ(t)$ in $d(W(t)/(\xi I(t)))$ and $dG(\Gamma^{\hat{\delta}}(t))$ derives the following relationship that involves the optimal investment strategy $\pi(t)$:

$$
\frac{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)} = G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\beta_3. \tag{47}
$$

By the principle of dynamic programming, it is convenient to express the investment as a function of initial variables at time 0:

$$
\frac{\pi^\top\sigma^\top - w(\sigma^I)^\top}{\xi I} = G'(z)z\beta_3, \tag{48}
$$

where $\pi = \pi(t) = \pi(0)$. Using the general solution $G(z)$ given in (33), a direct calculation of $G'(z)$ yields

$$
\begin{aligned}
G'(z) =& -\frac{1}{\gamma(\hat{A}+\delta)}z^{-1/\gamma-1} - \alpha_\delta^* B_\delta^* z^{-\alpha_\delta^*-1} + \frac{2\delta K}{||\beta_3||^2(1-\gamma)z^2}\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} \\
& - \frac{2\delta K\alpha_\delta(\alpha_\delta - 1)}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}z^{-\alpha_\delta-1}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}d\mu \\
& - \frac{2\delta K\alpha_\delta^*(\alpha_\delta^* - 1)}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}z^{-\alpha_\delta^*-1}\int_0^z \mu^{\alpha_\delta^*-2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}d\mu.
\end{aligned}
$$

Multiply $G'(z)$ by $\xi I z$ gives

$$
\begin{aligned}
\xi I G'(z)z =& -\frac{1}{\gamma(\hat{A}+\delta)}\xi I z^{-1/\gamma} - \alpha_\delta^*\xi I B_\delta^* z^{-\alpha_\delta^*} \\
& + \frac{2\delta K\xi I}{||\beta_3||^2(1-\gamma)z}\Big(G(z) - \frac{1}{\beta_1 + \frac{k}{\beta_1}} + \Big)^{1-\gamma} \\
& - \alpha_\delta \times \text{IP1} - \alpha_\delta^* \times \text{IP2},
\end{aligned}
$$

where IP1 and IP2 given in (39) are the first and second integral part of LNIS-induced precautionary savings. Note that $\xi I z^{-1/\gamma}$ of the first term in the above relationship is equivalent to the optimal consumption strategy from (42), as a result, $\xi I G'(z)z$ can be restated with (37) as the following:

$$\begin{aligned}\xi I G'(z)z = -\frac{1}{\gamma}\Big[&w + \frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*}\\ &- \frac{2\gamma}{||\beta_3||^2(1-\gamma)z}\xi I\delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}\\ &+ (\gamma\alpha_\delta - 1)\times \text{IP1} + (\gamma\alpha_\delta^* - 1)\times \text{IP2}\Big].\end{aligned}$$

Note that

$$\begin{aligned}&- \frac{2\gamma}{||\beta_3||^2(1-\gamma)z}\xi I\delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}\\ &= -\frac{2\gamma}{||\beta_3||^2 z}\xi I\delta K\frac{\Big(\frac{w}{\xi I} + \frac{k}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\\ &= -\frac{2\gamma}{||\beta_3||^2 z}(\xi I)^\gamma\delta K\frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\\ &= -\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}c(t)^\gamma\\ &= -\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}.\end{aligned}$$

Therefore, we derive the optimal investment strategy from (48):

$$\begin{aligned}\pi(t) &= \sigma^{-1}(\beta_3)^\top\xi I G'(z)z + \sigma^{-1}\sigma^I w\\ &= \sigma^{-1}(\gamma\sigma^I - \theta)\xi I G'(z)z + \sigma^{-1}\sigma^I w\\ &= \frac{1}{\gamma}\sigma^{-1}\theta w\\ &\quad + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*}\\ &\quad - \frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}\\ &\quad + (\gamma\alpha_\delta - 1)\times \text{IP1} + (\gamma\alpha_\delta^* - 1)\times \text{IP2}\Big].\end{aligned}$$

Until now, we have derived the optimal consumption and investment strategies, $c(t)$ and $\pi(t)$, in closed-form in (37) and (38), respectively, together with the income-shock-adjusted intensity $\hat{\delta}$ in

(27). Following Karatzas *et al.* (1991), the optimality would be verified if the wealth process $W(t)$ was self financed by $c(t)$ and $\pi(t)$. The term of dt of $dG(\Gamma^{\hat{\delta}}(t))$ in (46) is rewritten as

$$
\begin{aligned}
&-G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{(\beta_1^{\hat{\delta}}-\beta_2)+(1-\gamma)\beta_3\sigma^I\}+\frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2\\
&=\frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t))+\{\beta_2-\beta_1-\hat{\delta}+\delta-(1-\gamma)\beta_3\sigma^I\}\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\\
&=\frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t))+(\beta_2+\delta-\beta_1)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))-(1-\gamma)\beta_3\sigma^I\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\\
&\quad-\Big(G(\Gamma^{\hat{\delta}}(t))-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}\frac{\delta K}{\Gamma^{\hat{\delta}}(t)}\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\\
&=\frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t))+(\beta_2+\delta-\beta_1)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))-\delta K\Big(G(\Gamma^{\hat{\delta}}(t))-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{-\gamma}G'(\Gamma^{\hat{\delta}}(t))\\
&\quad-(1-\gamma)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\beta_3\sigma^I\\
&=-||\beta_3||^2\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))+\beta_1G(\Gamma^{\hat{\delta}}(t))-\Gamma^{\hat{\delta}}(t)^{-1/\gamma}-(1-\gamma)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\beta_3\sigma^I\\
&=-\frac{\pi(t)^\top\sigma^\top-W(t)(\sigma^I)^\top}{\xi I(t)}(\beta_3)^\top+\beta_1\Big(\frac{W(t)}{\xi I(t)}+\frac{1}{\beta_1}\Big)-\frac{c(t)}{\xi I(t)}-(1-\gamma)\frac{\pi(t)^\top\sigma^\top-W(t)(\sigma^I)^\top}{\xi I(t)}\sigma^I\\
&=\frac{1}{\xi I(t)}\Big[\{(\sigma^I)^\top(\beta_3)^\top+\beta_1\}W(t)-c(t)+\xi I(t)-\pi(t)^\top\sigma^\top(\beta_3)^\top-(1-\gamma)\pi(t)^\top\sigma^\top\sigma^I+(1-\gamma)||\sigma^I||^2W(t)\Big]\\
&=\frac{1}{\xi I(t)}\Big[\{(\sigma^I)^\top(\gamma\sigma^I-\theta)+r-\mu^I+(\sigma^I)^\top\theta\}W(t)-c(t)+\xi I(t)-\pi(t)^\top\sigma^\top(\gamma\sigma^I-\theta)\\
&\qquad-(1-\gamma)\pi(t)^\top\sigma^\top\sigma^I+(1-\gamma)||\sigma^I||^2W(t)\Big]\\
&=\frac{1}{\xi I(t)}\Big[\{r-\mu^I+\gamma||\sigma^I||^2\}W(t)-c(t)+\xi I(t)+\pi(t)^\top\{\mu-r\mathbf{1}-\sigma^\top\sigma^I\}\Big].
\end{aligned}
$$

where the second equality derives when $\hat{\delta}$ in (27) substituted in, the fourth equality derives from the differential equation in (28), the fifth equality derives from $||\beta_3||^2=\beta(\beta_3)^\top$, (41), (44), and (47). This shows that each term of dt in $d(W(t)/(\xi I(t)))$ and $dG(\Gamma^{\hat{\delta}}(t))$ are exactly the same, as a result, the wealth process $W(t)$ is self financed by the optimal consumption strategy $c(t)$ and the optimal investment strategy $\pi(t)$ with the income-shock-adjusted intensity $\hat{\delta}$ in (27). **Q.E.D.**

Theorem 3.1 allows us to obtain the resulting optimal (riskless) savings by measuring the wedge between total wealth (financial wealth+human capital) and the sum of consumption and investment. Specifically, we identify and quantify three different optimal savings motives in the following Corollary: (i) PIH-implied optimal savings, (ii) Borrowing-constraints-induced optimal savings, and (iii) LNIS-induced optimal savings.

Corollary 3.1. *We identify and quantify three different optimal savings motives as follows.*

(i) PIH-implied optimal savings

$$= \Big(1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta\Big)\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^{I}\frac{\xi I}{\beta_1}.$$

(ii) Borrowing-constraints-induced optimal savings

$$= \textit{PIH-implied optimal savings}$$
$$+ \Big(\hat{A} - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^{I})(\gamma\alpha_0^* - 1)\Big)\xi I B_0^* z^{-\alpha_0^*}.$$

(iii) LNIS-induced optimal savings

$$= \textit{PIH-implied optimal savings} + \textit{LNIS-PS},$$

where LNIS-PS is the LNIS-induced precautionary savings given in (40).

Proof. Without the time-varying borrowing constraints (17) and the LNIS, i.e., when $B_0^* = 0$ and $\delta = 0$, the agent's optimal consumption strategy (37) can be rewritten by the following Friedman's (1957) PIH:

$$c(t) = \hat{A}\Big(w + \frac{\xi I}{\beta_1}\Big).$$

Also, the classic Merton (1969, 1971) investment rule can be revisited:

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^{I}\frac{\xi I}{\beta_1},$$

which comes from (38) when $\sigma_I > 0$ and $\delta = 0$, i.e., with output uncertainty but without the LNIS. Friedman's PIH-implied optimal savings are then defined as total wealth minus the sum of consumption and investment, so that

$$\begin{aligned}
&\text{PIH-implied optimal savings} \equiv w + \frac{\xi I}{\beta_1} - c(t) - \pi(t)\\
&= w + \frac{\xi I}{\beta_1} - \hat{A}\Big(w + \frac{\xi I}{\beta_1}\Big) - \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^{I}\frac{\xi I}{\beta_1}\\
&= \Big(1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta\Big)\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^{I}\frac{\xi I}{\beta_1},
\end{aligned}$$

which is the PIH-implied optimal savings in Corollary 3.1.

With the time-varying borrowing constraints but the LNIS, i.e., when $B_0^* > 0$ and $\delta = 0$, the agent's optimal consumption strategy (37) can be restated as

$$c(t) = \hat{A}\Big(w + \frac{\xi I}{\beta_1} - \xi I B_0^* z^{-\alpha_0^*}\Big).$$

The individual whose borrowing is constrained by (17) derives the following optimal investment strategy:

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1} + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\xi I B_0^* z^{-\alpha_0^*}. \tag{49}$$

The borrowing-constraints-induced optimal savings are then given by

$$\begin{aligned}
&\text{Borrowing-constraints-induced optimal savings}\\
&\equiv w + \frac{\xi I}{\beta_1} - c(t) - \pi(t)\\
&= w + \frac{\xi I}{\beta_1} - \hat{A}\Big(w + \frac{\xi I}{\beta_1} - \xi I B_0^* z^{-\alpha_0^*}\Big)\\
&\quad - \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1} - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\xi I B_0^* z^{-\alpha_0^*}\\
&= \Big(1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta\Big)\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1}\\
&\quad + \Big(\hat{A} - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\Big)\xi I B_0^* z^{-\alpha_0^*},
\end{aligned}$$

which is the borrowing-constraints-induced optimal savings in Corollary 3.1.

With both the time-varying borrowing constraints and the LNIS, the agent's LNIS-induced optimal savings are defined as total wealth minus the sum of consumption and investment which are given in Theorem 3.1:

$$\begin{aligned}
&\text{LNIS-induced optimal savings}\\
&\equiv w + \frac{\xi I}{\beta_1} - c(t) - \pi(t)\\
&= \Big(w + \frac{\xi I}{\beta_1}\Big) - (\hat{A} + \delta)\Big(w + \frac{\xi I}{\beta_1} - \xi I B_\delta^* z^{-\alpha_\delta^*} - \text{IP}\Big)\\
&\quad - \frac{1}{\gamma}\sigma^{-1}\theta w - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*}\\
&\quad - \frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c(t)^{-\gamma} + (\gamma\alpha_\delta - 1)\times\text{IP1} + (\gamma\alpha_\delta^* - 1)\times\text{IP2}\Big]\\
&= \text{PIH-implied optimal savings} + \text{LNIS-PS},
\end{aligned}$$

which is the LNIS-induced optimal savings given in Corollary 3.1. **Q.E.D.**

Discussion on Optimal Consumption/Savings. Without the time-varying borrowing constraints (17) and the LNIS, i.e., when $B_0^* = 0$ and $\delta = 0$, the agent's optimal consumption strategy (37) can be rewritten by the following Friedman's (1957) PIH:

$$c(t) = \hat{A}\Big(w + \frac{\xi I}{\beta_1}\Big),$$

which means that the agent's consumption can be annuitized from her total available resources. Further, the marginal propensity to consume out of financial wealth is constant implying that regardless of wealth levels, the agent's optimal consumption to total wealth ratio is well maintained at constant rate.

The classic Merton (1969, 1971) investment rule can be revisited:

$$\pi(t) = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1}, \tag{50}$$

which comes from (38) when $\sigma_I > 0$ and $\delta = 0$, i.e., with output uncertainty but without the LNIS. The first term on the right hand side of (50) represents the mean-variance asset allocation and the second one represents the demand for hedging (or the intertemporal hedging component) against the output uncertainty.

The PIH-implied optimal savings show that the marginal propensity to save (MPS) out of financial wealth is $1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta$, which implies that with respect to one unit increase of wealth the constant portion of the agent's extra money aside from consumption portion $\hat{A}$ and investment portion $\frac{1}{\gamma}\sigma^{-1}\theta$ is to be optimally put into her riskless savings. This savings strategy, however, has been at odds with empirical evidence (Federal Reserve report, 2017; EU-SILC 2017) in that the agent's savings are too small to address the financial challenges on her future consumption.

In addition to the PIH-implied optimal savings, the extra terms on the right hand side of borrowing-constraints-induced optimal savings in 3.1 represent additional precautionary savings motive for avoiding being binded by the borrowing constraints given in (17). The MPS out of financial wealth becomes larger or smaller than that of the PIH-implied optimal savings by the amount of $-\alpha_0^*\big(\hat{A} - \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\big)\xi I B_0^* z^{-\alpha_0^*} z'(w)$.[3] Here, $-\frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)(\gamma\alpha_0^* - 1)\xi I B_0^* z^{-\alpha_0^*}$ captures endogenous

[3] $z(w)$ is a dual function of financial wealth w and known to be decreasing and convex implying $z'(w) < 0$.

adjustments in the savings with benchmarking. When the benchmark is taken to be aggregate output, $\sigma^I\theta^{-1}$ can quantify the sensitivity of the benchmark (or the aggregate output) to economic conditions, and $1/\gamma$ can quantify the sensitivity of the Merton policy (or the stock investment) to economic conditions.[4] Normally, the aggregate output becomes less sensitive to economic conditions than the stock investment, i.e., $\sigma^I\theta^{-1} < 1/\gamma$.[5] Hence, the MPS is smaller than that of the PIH-implied optimal savings.

The smaller MPS with borrowing constraints implies that with respect to one unit decrease of wealth the agent is inclined to less reduce her savings amount, as she is responsible for maintaining her wealth to be larger than the time-varying constraint $-L(t)$ given in (17) in all states. Interestingly, the MPS further decreases as wealth decumulates (as a result, z becomes larger). This shows that the borrowing-constrained precautionary savings motive has a progressively more stronger impact on the agent's total savings when wealth is small, thereby further increasing demand for savings at low levels of wealth in the preparation against market downturns.

We generalize Friedman's PIH-implied optimal savings with the LNIS-induced precautionary savings ($LNIS - PS$). The MPS of the $LNIS - PS$ out of financial wealth is

$$\begin{aligned}\frac{\partial(LNIS-PS)}{\partial w} =& -\delta - \alpha_\delta^*\Big(\hat{A}+\delta-\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^I)(\gamma\alpha_\delta^*-1)\Big)\xi I B_\delta^* z^{-\alpha_\delta^*-1}z'(w)\\ &+\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^I)\frac{2\gamma}{||\beta_3||^2}\delta K\Big(w+\frac{k\xi I}{\beta_1}\Big)^{-\gamma}\Big/c(t)^{-\gamma}\\ &-\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^I)\frac{2\gamma}{||\beta_3||^2z^2}(\xi I)^\gamma\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}z'(w)\\ &+\Big(\hat{A}+\delta-\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^I)(\gamma\alpha_\delta-1)\Big)\frac{\partial(IP1)}{\partial z}z'(w)\\ &+\Big(\hat{A}+\delta-\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^I)(\gamma\alpha_\delta^*-1)\Big)\frac{\partial(IP2)}{\partial z}z'(w),\end{aligned}$$

where

$$\begin{aligned}\frac{\partial(IP1)}{\partial w} =& -\frac{2\delta K\alpha_\delta(\alpha_\delta-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}z^{-\alpha_\delta-1}\int_0^z\mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\\ &+\frac{2\delta K(\alpha_\delta-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)z^2}\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}>0,\end{aligned}$$

[4]Basak *et al.* (2006) have adopted these quantities in their analysis for the risk management with benchmarking.

[5]In an economy in which the aggregate output reacts more to changes in economic conditions than the stock investment, i.e., when i.e., if $\sigma^I\theta^{-1} > 1/\gamma$, the MPS can be larger than that of the PIH-implied optimal savings in some cases. In this case, the individual earnings no longer act as a substitute for the implicit cash holdings, rather these earnings behave like a stochastic stream which may be riskier than the stock investment. The individual's optimal choice would, thus, be to invest more in the stock market as in Cocco *et al.* (2005).

$$
\begin{aligned}
\frac{\partial(IP2)}{\partial w} = &-\frac{2\delta K\alpha_\delta^*(\alpha_\delta^*-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*-1}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\
&-\frac{2\delta K(\alpha_\delta^*-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)z^2}\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma} > 0.
\end{aligned}
$$

Compared to the borrowing-constraints-induced optimal savings,[6] the extra first two terms involving

$$
\delta K\Big(w+\frac{k\xi I}{\beta_1}\Big)^{-\gamma}\Big/ c(t)^{-\gamma}
$$

and

$$
\delta K\frac{\Big(w+\dfrac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}
$$

on the right hand side of the MPS further increase the MPS, as those two terms capture a hedging demand for risk diversification that turns out to decrease the optimal (riskless) savings and instead increase the optimal investment in the stock market as in (38) by the amount proportional to the ratio of the agent's utility value from total available financial resources for satisfying consumption needs in the LNIS, $\delta K(w + k\xi I/\beta_1)^{1-\gamma}/(1-\gamma)$, to the marginal utility of one extra unit of consumption, $c^{-\gamma} = (c^{1-\gamma}/(1-\gamma))'$. This ratio does imply that a larger amount of stock investment is required for higher consumption needs in the LNIS.

Interestingly, in addition to such a risk diversification demand, the agent would show a savings demand for precautionary reasons in the event of the LNIS. Indeed, the integral parts $IP1$ and $IP2$ of $LNIS - PS$ can play a role to further decrease the MPS and thus, the agent tends to less reduce her savings amount with respect to one unit decrease of wealth. Rather, the agent reduces her consumption amount by $(\hat{A}+\delta)\times IP$ as in (37) and increases her savings amount as in the LNIS-induced optimal savings (or in the LNIS-induced precautionary savings) in Corollary 3.1.

Given the differences between the PIH-implied optimal savings, borrowing-constraints-induced optimal savings, and LNIS-induced optimal savings as we have analyzed so far, it is worth to thoroughly investigate a role of the LNIS in the following two points: (i) what and how the extra LNIS-induced precautionary savings motive would affect the general equilibrium interest rate, and (ii) such a savings movie could improve the equilibrium model's ability to match the equity premium and risk-free rate observed in the data.

[6] The LNIS-induced optimal savings reduce to the borrowing-constraints-induced optimal savings without consideration of the LNIS ($\delta = 0$).

4 Technical Details of a General Equilibrium Analysis

In the main manuscript, we have derived the general equilibrium quantities in the presence of the LNIS. We consider a simple exchange economy in the style of Lucas (1978). The economy is populated by a representative agent facing the LNIS. The agent is entitled to an aggregate endowment to be consumed in equilibrium and is assumed to trade a riskless bond and multiple risky stocks distributing the dividend. The returns to these assets adjust to represent a no-trade equilibrium. The risk-free interest rate, r, the constant mean vector, μ, and the constant nonsingular standard deviation matrix, σ, should be determined from the equilibrium conditions, as specified below:

Definition 4.1. *An equilibrium can be characterized as a collection of* (r, μ, σ) *and optimal strategies* $(c(t), \pi(t))$ *such that the consumption good, stock, and bond markets clear as*

$$\begin{aligned} &c(t) = I(t), \\ &\pi^j(t) = S^j(t), \;\; j = 1, ..., N, \\ &W(t) = \sum_{j=1}^{N} S^j(t), \end{aligned}$$

where N *is the number of risky stocks.*

The following proposition provides the unique state price density in the presence of the market risk and the income shock.

Proposition 4.1. *The unique state price density is given by*

$$\xi^{\hat{\delta}}(t) = \exp\Big\{ \ln\Big(\frac{\hat{\delta}}{\delta}\Big)\boldsymbol{1}_{\{\tau \le t\}} - (\hat{\delta} - \delta)t \Big\} H(t), \tag{51}$$

where

$$\hat{\delta} = \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z},$$

τ *is the arrival time of a Poisson shock,* $\boldsymbol{1}$ *is an indicator function that gives* 1 *if the Poisson shock occurs at time* t *and* 0 *otherwise,* $G(z)$ *satisfies the following differential equation:*

$$\begin{aligned} &-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ &\qquad + \beta_1 G(z) + \delta K \Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \;\; 0 < z < \overline{z}, \end{aligned} \tag{52}$$

with

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1}, G'(\overline{z}) = 0,$$

and the dynamics of $H(t)$ are given by

$$dH(t) = -H(t)\{rdt + \theta^\top dZ(t)\}, \quad H(0) = 1.$$

Proof. See 3. **Q.E.D.**

Given the unique state price density, the following proposition solves for the equilibrium state price density, and the equilibrium risk-free interest rate and the equilibrium Sharpe ratio.

Proposition 4.2. *The equilibrium state price density prior to the LNIS is given by: $t < \tau$,*

$$H(t) = \frac{1}{\lambda} e^{-(\beta - (\hat{\delta}(r) - \delta))t} (I(t))^{-\gamma}, \tag{53}$$

the equilibrium risk-free interest rate and the equilibrium Sharpe ratio prior to the LNIS are given by: $t < \tau$

$$r = \beta + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)(\sigma^I)^2 - (\hat{\delta}(r) - \delta) \tag{54}$$

and

$$\theta = \gamma\sigma^I, \tag{55}$$

respectively, where μ^I and σ^I represent the expected consumption growth rate and volatility of consumption growth rate, and the constant $\hat{\delta}(r)$ is determined by solving the following non-linear algebraic equation:

$$\hat{\delta}(r) = \left\{ \left(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))} \right) \Big/ \left(\frac{w}{\xi I} + \frac{k}{\beta_1(\hat{\delta}(r))} \right) \right\}^\gamma \{\beta_1(\hat{\delta}(r))\}^\gamma \delta K(r)$$

with

$$\beta_1(\hat{\delta}(r)) = \beta + (\gamma - 1)\mu^I - \frac{1}{2}\gamma(\gamma - 1)(\sigma^I)^2 - (\hat{\delta}(r) - \delta),$$

$$K(r) = \left\{ \frac{\gamma - 1}{\gamma} \left(r + \frac{\gamma(\sigma^I)^2}{2} \right) + \frac{\beta}{\gamma} \right\}^{-\gamma},$$

and the constant λ satisfies

$$E\Big[\int_0^\infty e^{-\hat\delta(r)t}H(t)\Big\{c(t)-\xi I(t)+\hat\delta(r)W(t)\Big\}dt\Big]=w, \tag{56}$$

$$c(t)=\Big(\lambda e^{(\beta+\delta-\hat\delta(r))t}H(t)\Big)^{-1/\gamma},\ \ W(t)=\Big(\lambda e^{(\beta+\delta-\hat\delta(r))t}H(t)\Big)^{-1/\gamma}\Big(\frac{\hat\delta(r)}{\delta}\Big)^{-1/\gamma}K^{1/\gamma}-\frac{k\xi I(t)}{\beta_1(\hat\delta(r))},$$

with (53), (54), and (55) substituted in.

Proof. We can expect that the LNIS may result in jumps in the state price densities and the equilibrium securities accordingly. We, thus, need to change the bond and stock price dynamics. We can posit that the price dynamics are still unchanged, but at time τ we allow for an extra jump component $\psi\mathbf{1}_{\{\tau\le t\}}$ in the price dynamics. Here, $\mathbf{1}_{\tau\le t}$ is a (right-continuous) step function so that $d\mathbf{1}_{\tau\le t}$ is a measure assigning unit mass to time τ. The jump coefficient ψ is an $\mathcal{F}_\tau$-measurable random variable associated with the price jumps by

$$\begin{aligned}\psi&=\ln\Big(H(\tau-)/H(\tau)\Big)\\&=\ln\Big(B(\tau-)/B(\tau)\Big)\\&=\ln\Big(S^j(\tau-)/S^j(\tau)\Big),\ \ j=1,2,...,N,\end{aligned}$$

where $H(\tau-)$, $B(\tau-)$, $S^j(\tau-)$ are the left limits at τ. Notice that since $\mathcal{F}_{\tau-}=\mathcal{F}_\tau$, the jump coefficient ψ in the state price densities and the security prices must be the same, otherwise there is an arbitrage on these jumps. Therefore, the discounted bond and stock prices and wealth are all still continuous at all times.

The optimal consumption strategy prior to the LNIS derives from (25): $t<\tau$,

$$c(t)=\Big(\lambda e^{(\beta+\delta-\hat\delta(r))t}H(t)\Big)^{-1/\gamma},$$

where the constant λ should satisfy

$$E\Big[\int_0^\infty e^{-\hat\delta(r)t}H(t)\Big\{c(t)-\xi I(t)+\hat\delta(r)W(t)\Big\}dt\Big]=w$$

with the following optimal wealth process prior to the LNIS: $t < \tau$,

$$W(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta}(r))t}H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}(r)}{\delta}\Big)^{-1/\gamma}K^{1/\gamma} - \frac{k\xi I(t)}{\beta_1(\hat{\delta}(r))},$$

and $\hat{\delta}(r)$ prior to the LNIS is given by Proposition 4.1 as follows: $t < \tau$,

$$\hat{\delta}(r) = \Big(G(z) - \frac{1}{\beta_1(\hat{\delta}(r))} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)^{-\gamma}\frac{\delta K}{z}, \tag{57}$$

where $G(z)$ solves the differential equation in (52) with

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}, G'(\overline{z}) = 0.$$

According to the clearing condition of consumption good, $c(t) = I(t)$, the equilibrium state price density $H(t)$ prior to the LNIS follows: $t < \tau$,

$$H(t) = \frac{1}{\lambda}e^{-(\beta-(\hat{\delta}(r)-\delta))t}I(t)^{-\gamma}. \tag{58}$$

Applying Itô's formula to the both sides of (58) prior to the LNIS i.e., for any $t < \tau$,

$$\begin{aligned}&-H(t)\{rdt + \theta^\top dZ(t)\}\\&= -H(t)\Big\{\Big(\beta - (\hat{\delta}(r)-\delta) + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2\Big)dt + \gamma(\sigma^I)^\top dZ(t)\Big\},\end{aligned}$$

where we do not consider in general equilibrium the Poisson jump term of $I(t)$ because its effects are reflected by $\hat{\delta}(r) - \delta$ in the drift term on the right hand side of the equation, which results from the generalized state price densities given in (53). Equating each term of dt and $dZ(t)$ gives the equilibrium risk-free interest rate and the equilibrium Sharpe ratio as stated in (54) and (55).

When the LNIS occurs, i.e., at $t = \tau$, the jump-size parameter is determined in equilibrium by measuring the anticipated downward jump in $H(t)$ at time $t = \tau$. In this case, the equilibrium consumption price $c(t)$ jumps down to correspond to the downward jump in aggregate demand for consumption at time τ due to reduced total resources caused by the LNIS. The equilibrium state price density $H(t)$ in the LNIS follows: $t \geq \tau$,

$$H(t) = \frac{1}{\tilde{\lambda}}e^{-\beta t}(kI(t))^{-\gamma},$$

where the constant $\tilde{\lambda}$ should satisfy

$$E\Big[\int_0^\infty H(t)\{c(t) - k\xi I(t)\}dt\Big] = w.$$

The jump-size parameter ψ would be, thus, determined by

$$\begin{aligned}\psi &= \ln\Big(H(\tau-)/H(\tau)\Big)\\ &= \ln\Big\{\Big(\frac{1}{\lambda}e^{-(\beta-(\hat{\delta}(r)-\delta))\tau}I(\tau)^{-\gamma}\Big)\Big/\Big(\frac{1}{\tilde{\lambda}}e^{-\beta\tau}(kI(\tau))^{-\gamma}\Big)\Big\}\\ &= \ln\Big(\frac{\tilde{\lambda}}{\lambda}e^{(\hat{\delta}(r)-\delta)\tau}k^{\gamma}\Big)\end{aligned}$$

With the equilibrium quantities (54) and (55), and the clearing conditions of stock and bond markets, the differential equation in (52) has a solution in closed-form:

$$G(z) = \frac{1}{\beta_1(\hat{\delta}(r))}z^{-1/\gamma}.$$

By substituting the solution for $G(z)$ in (57), $\hat{\delta}(r)$ is determined by

$$\hat{\delta}(r) = \Big(\frac{1}{\beta_1(\hat{\delta}(r))}z^{-1/\gamma} - \frac{1}{\beta_1(\hat{\delta}(r))} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)^{-\gamma}\frac{\delta K}{z}. \tag{59}$$

Recall the relationship (43)

$$w = \xi I\Big(G(z) - \frac{1}{\beta_1(\hat{\delta}(r))}\Big),$$

accordingly,

$$\frac{1}{\beta_1(\hat{\delta}(r))}z^{-1/\gamma} = \frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))},$$

or equivalently,

$$z^{-1} = \{\beta_1(\hat{\delta}(r))\}^{\gamma}\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)^{\gamma}.$$

As a result, the equation (59) reduces to

$$\hat{\delta}(r) = \Big\{\Big(\frac{w}{\xi I} + \frac{1}{\beta_1(\hat{\delta}(r))}\Big)\Big/\Big(\frac{w}{\xi I} + \frac{k}{\beta_1(\hat{\delta}(r))}\Big)\Big\}^{\gamma}\{\beta_1(\hat{\delta}(r))\}^{\gamma}\delta K$$

with

$$\begin{aligned}\beta_1(\hat{\delta}(r)) &= r - \mu^I + (\sigma^I)^\top \theta \\ &= \beta + \gamma\mu^I - \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2 - (\hat{\delta}(r) - \delta) - \mu^I + (\sigma^I)^\top \theta \\ &= \beta + (\gamma - 1)\mu^I - \frac{1}{2}\gamma(\gamma - 1)||\sigma^I||^2 - (\hat{\delta}(r) - \delta),\end{aligned}$$

where the second equality derives from the substitution of (54) and the third equality comes from the substitution of (55). This completes the proof. **Q.E.D.**

5 Technical Details of Quantitative Analysis

Lower Bound of Wealth. In terms of empirical reality, we can relate the lower bound $-L(t)$ with $L(0) = L$ of wealth in (17) to a tightening of credit, which is empirically plausible as Survey of Consumer Finances (2017) states as follows:

> In 2016, 20.8 percent of families were considered credit constrained – those who reported being denied credit in the past year, as well as those who did not apply for credit for fear of being denied in the past year (Survey of Consumer Finances, 2017).

Borrowing against human capital is constrained fully or partly. Thus, the extent to which credit is tightened, i.e., the level of lower bound of wealth becomes a real consideration.

Aggregate earnings are assumed to be given by a constant income stream. Specifically, the earnings are given by $\epsilon \equiv \xi I$ over the life cycle. Then, the lower bound $-L(t)$ with $L(0) = L$ of wealth in (17) can be empirically plausible using the following relationship:

$$L = \omega \frac{r + \nu + \delta k}{(r + \nu + \delta)(r + \nu)} \epsilon, \quad \text{for } 0 \leq \omega < 1, \tag{60}$$

where $\epsilon = \xi I$ represents a constant stream of earnings over the life cycle, ω represents the extent to which credit is tightened and $\nu > 0$ is the agent's constant mortality intensity when the time to death is distributed with an exponential distribution.[7] The utility related to death is normalized as zero.[8]

[7]The constant mortality rate assumption is made for parsimony of the model, helping explore horizon-dependent polices in the simplest possible economic environment. The derived model predictions are consistent with the typical life-cycle advice. A more realistic model would allow for a Gompertz force of mortality, which is quite relevant to the actuarial literature.

[8]On account of this normalization, we do not consider motive for bequest. The presence of bequest motive is expected to reinforce the negative impacts of the LNIS.

The relationship (60) shows that the value of human capital depreciates relative to that predicted by Friedman (1957). Without the LNIS (i.e., $\delta = 0$), the lower bound reduces to $\epsilon/(r+\nu)$ as ω approaches one (i.e., when credit tightening does not occur). With the LNIS (i.e., $\delta > 0$), the lower bound is always larger than $-\epsilon/(r+\nu)$, which is more credit tightened than without the LNIS.

Now, we will verify the relationship (60). Following Friedman (1957) and Hall (1978), the present value of future earnings is calculated by their discounted value at the risk-free interest rate. More specifically, with the stochastic life-cycle earnings $\epsilon(t)$ given by

$$\epsilon(t) = \begin{cases} \epsilon, & \text{if } 0 \le t < \tau \wedge \tau^{\nu}, \\ k\epsilon, & \text{if } \tau \le t < \tau^{\nu}, \\ 0, & \text{if } t \ge \tau^{\nu}, \end{cases}$$

the present value of future life-cycle earnings follows

$$\begin{aligned}
&E\Big[\int_0^{\tau\wedge\tau^{\nu}} e^{-rt}\epsilon(t)dt + e^{-r(\tau\wedge\tau^{\nu})}\int_{\tau\wedge\tau^{\nu}}^{\infty} e^{-r(t-\tau\wedge\tau^{\nu})}\epsilon(t)dt\Big] \\
&= E\Big[\int_0^{\infty} \nu e^{-\nu s}\int_0^{\tau\wedge s} e^{-rt}\epsilon(t)dtds + \int_0^{\infty} \nu e^{-\nu s} e^{-r(\tau\wedge s)}\int_{\tau\wedge s}^{\infty} e^{-r(t-\tau\wedge s)}\epsilon(t)dtds\Big] \\
&= E\Big[\int_0^{\tau} \nu e^{-\nu s}\int_0^{s} e^{-rt}\epsilon dtds + \int_{\tau}^{\infty} \nu e^{-\nu s}\int_0^{\tau} e^{-rt}\epsilon dtds \\
&\qquad + \int_0^{\tau} \nu e^{-\nu s}e^{-rs}\int_s^{\infty} e^{-r(t-s)}0dtds + \int_{\tau}^{\infty} \nu e^{-\nu s}e^{-r\tau}\int_{\tau}^{s} e^{-r(t-\tau)}k\epsilon dtds\Big] \\
&= E\Big[\int_0^{\tau} e^{-rt}\epsilon\int_t^{\tau} \nu e^{-\nu s}dsdt + \int_0^{\tau} e^{-rt}\epsilon\int_{\tau}^{\infty} \nu e^{-\nu s}dsdt + \int_{\tau}^{\infty} e^{-rt}k\epsilon\int_t^{\infty} \nu e^{-\nu s}dsdt\Big] \\
&= E\Big[\int_0^{\tau} e^{-rt}\epsilon\int_t^{\infty} \nu e^{-\nu s}dsdt + \int_{\tau}^{\infty} e^{-rt}k\epsilon\int_t^{\infty} \nu e^{-\nu s}dsdt\Big] \\
&= E\Big[\int_0^{\tau} e^{-(r+\nu)t}\epsilon dt + \int_{\tau}^{\infty} e^{-(r+\nu)t}k\epsilon dt\Big] \\
&= \int_0^{\infty} \delta e^{-\delta s}\int_0^{s} e^{-(r+\nu)t}\epsilon dtds + \int_0^{\infty} \delta e^{-\delta s}\int_s^{\infty} e^{-(r+\nu)t}k\epsilon dtds \\
&= \int_0^{\infty} e^{-(r+\nu)t}\epsilon\int_t^{\infty} \delta e^{-\delta s}dsdt + \int_0^{\infty} e^{-(r+\nu)t}k\epsilon\int_0^{t} \delta e^{-\delta s}dsdt \\
&= \int_0^{\infty} e^{-(r+\nu+\delta)t}\epsilon dt + \int_0^{\infty} e^{-(r+\nu)t}k\epsilon(1-e^{-\delta t})dt \\
&= \frac{1}{r+\nu+\delta}\epsilon + \frac{1}{r+\nu}k\epsilon - \frac{1}{r+\nu+\delta}k\epsilon \\
&= \frac{r+\nu+\delta k}{(r+\nu+\delta)(r+\nu)}\epsilon.
\end{aligned}$$

$w \setminus \omega$	$\delta = 0$				$\delta = 0.07$				$\delta = 0.08$			
	0%	5%	10%	20%	0%	5%	10%	20%	0%	5%	10%	20%
1	1.0922	1.2090	1.2966	1.4336	0.7261	0.7200	0.7232	0.7281	0.7077	0.7033	0.7060	0.7098
10	1.8159	1.8646	1.9104	1.9948	1.1919	1.1809	1.1808	1.1810	1.1691	1.1578	1.1575	1.1586
20	2.4041	2.4402	2.4750	2.5409	1.6679	1.6538	1.6538	1.6539	1.6406	1.6285	1.6285	1.6285
30	2.9396	2.9698	2.9991	3.0552	2.1348	2.1183	2.1185	2.1185	2.1040	2.0915	2.0918	2.0910
40	3.4507	3.4772	3.5031	3.5528	2.5966	2.5787	2.5789	2.5789	2.5636	2.5507	2.5510	2.5501
50	3.9476	3.9715	3.9949	4.0400	3.0552	3.0365	3.0368	3.0368	3.0210	3.0075	3.0078	3.0069

$w \setminus \omega$	$\delta = 0.09$				$\delta = 0.10$			
	0%	5%	10%	20%	0%	5%	10%	20%
1	0.6862	0.6887	0.6909	0.6945	0.6748	0.6815	0.6784	0.6814
10	1.1396	1.1397	1.1397	1.1398	1.1223	1.1094	1.1233	1.1237
20	1.6071	1.6071	1.6072	1.6071	1.5894	1.5913	1.5890	1.5888
30	2.0680	2.0681	2.0681	2.0681	2.0498	2.0618	2.0489	2.0486
40	2.5260	2.5260	2.5260	2.5260	2.5069	2.5208	2.5059	2.5057
50	2.9820	2.9821	2.9821	2.9821	2.9620	2.9738	2.9613	2.9611

Table 1: **Optimal consumption amount for various credit tightening scenarios and intensity values of the large, negative shock.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate).

Therefore, the lower bound L at time 0 in (60) derives from the multiplication of the present value stated above by exogenously given $\omega \in [0, 1)$ that represents the extent to which credit is tightened.

6 Further Numerical Results

Effects of Credit Tightening. Credit tightening affects the optimal consumption (Table 1) and investment (Table 2) strategies. Tightening of credit by decreasing ω makes individuals reduce their consumption amount; this response is especially significant for poor people, so their consumption smoothing is more difficult than for wealthy people. The effects of the LNIS worsens the situation for poor people. Given the significant downward jump in income in the aftermath of the LNIS, the poor people who are credit tightened would have difficulty to secure extra savings to finance their consumption needs. Hence, the consumption amount could fall further with the joint effects caused by the credit tightening and the income shock. Those effects also reduce the risky investment amount; this result is similar to the observation in the optimal consumption amount.

Hedging Demand. The amount of hedging demands differs between $\delta = 0$ and $\delta > 0$, and the difference increases as wealth w increases (Figure 1). This result implies that even sufficiently large wealth does not appropriately absorb the negative effects of the LNIS, and thereby amplifies the negative effects of background risk on risky investment compared to the positive effects of risk diversification.[9]

[9] There are two opposing motives on risky investment: a precautionary savings motive that reduces investment and a

$w \setminus \omega$	$\delta = 0$				$\delta = 0.07$				$\delta = 0.08$			
	0%	5%	10%	20%	0%	5%	10%	20%	0%	5%	10%	20%
1	4.0426	6.1272	7.6885	10.1335	5.2955	5.6019	5.8596	6.2650	5.3110	5.5845	5.8131	6.1575
10	14.2285	15.0976	15.9148	17.4194	12.3362	12.2797	12.8383	12.2981	12.2003	12.1282	12.1269	12.1598
20	21.6927	22.3378	22.9587	24.1342	17.7237	17.6876	17.6797	17.6806	17.5918	17.4760	17.4635	17.5022
30	28.2186	28.7577	29.2809	30.2818	23.0102	22.9245	22.9209	22.9207	22.8117	22.6988	22.6947	22.7050
40	34.3084	34.7818	35.2955	36.1304	28.2267	28.0941	28.0950	28.0951	27.9683	27.8593	27.8622	27.8523
50	40.1449	40.5720	40.9892	41.7945	33.3908	33.2257	33.2293	33.2298	33.0902	32.9808	32.9875	32.9675

$w \setminus \omega$	$\delta = 0.09$				$\delta = 0.10$			
	0%	5%	10%	20%	0%	5%	10%	20%
1	5.3298	5.5553	5.7509	6.0713	5.3526	5.7242	5.7158	5.9980
10	12.0225	12.0263	12.0312	12.0399	11.8807	11.6225	11.9189	11.9382
20	17.3480	17.3468	17.3471	17.3471	17.1623	16.6403	17.2001	17.2086
30	22.5196	22.5197	22.5197	22.5194	22.3573	22.2349	22.3573	22.3544
40	27.6439	27.6450	27.6452	27.6454	27.4934	27.6851	27.4720	27.4644
50	32.7435	32.7451	32.7453	32.7456	32.5915	32.9486	32.5630	32.5543

Table 2: **Optimal investment amount for various credit tightening scenarios and intensity values of the large, negative shock.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate).

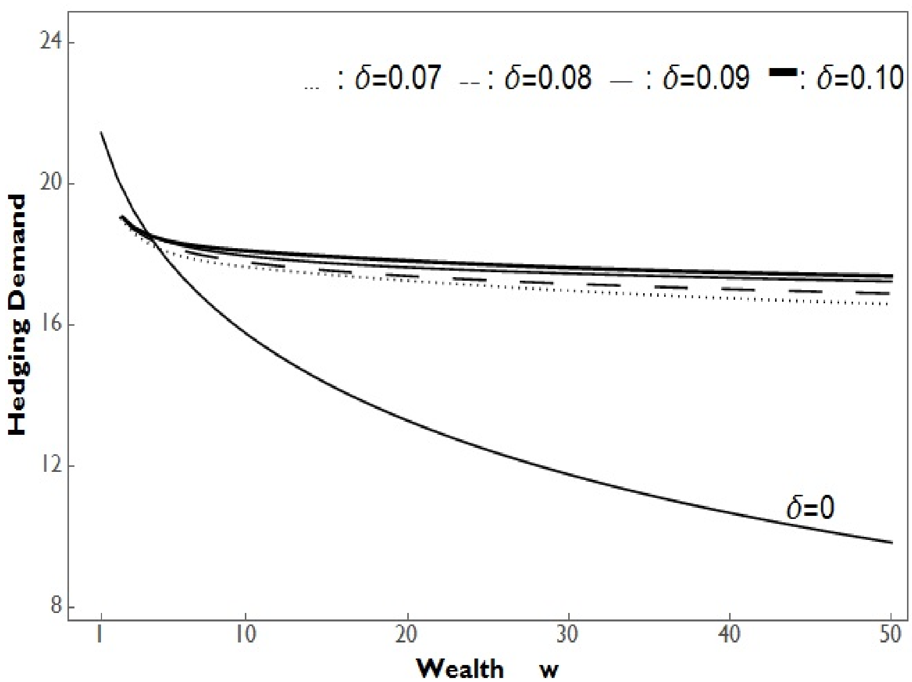

Figure 1: **Hedging demand.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate).

Data Description. The data on wealth and income are from family net worth and before tax family income by selected characteristics of families in the SCF for the period of 1995-2010. We sort the

risk diversification motive that rises investment.

	Age				
Percentile of Net Worth	35-44	45-54	55-64	65-74	75-80
0-25	6.0	5.7	5.5	1.3	0.5
25-49.9	10.7	10.4	9.7	3.4	2.0
50-74.9	14.8	14.6	14.2	6.2	4.1
75-89.9	17.7	17.6	17.3	9.1	6.7
90-100	20.1	20.0	19.8	12.0	9.4
all	14.8	14.6	14.2	4.1	4.1

Table 3: **The optimal proportion of total wealth invested in stocks.**Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), and $k = 0.2$ (recovery rate). The values of mortality rate, ν, are adjusted following age of head given in the SCF, assuming that people die on average at the age of 80. An initial endowment of financial wealth, w, and a constant stream of labor income per annum, $\epsilon = \xi I$, are calibrated via the normalized cash-on-hand, i.e., the net worth normalized by income from the SCF for the period of 1995-2010. Note that the data period is chosen for including the 2007-2009 Great Recession in the U.S., when many people have experienced the unprecedented largest reductions in their consumption and unemployment.

family net worth and before tax family income into age groups and percentile of net worth, and use them to compute the normalized cash-on-hand, which is the ratio of net worth to income. We find from the data that the family net worth (before tax family income) by age groups shows a hump-shaped profile; until age of 55-64 (45-54) the net worth (the family income) increases with age, but subsequently falls with age on average. The net worth and the family income increase with an increase in percentile of net worth.

Calibration. To obtain empirically plausible implications on investment, we carefully choose the parameter values for mortality rate and aggregate earnings by using the Survey of Consumer Finances (SCF) data. We adjust the values of mortality rate, ν, according to age of head given in the SCF. We assume that people die on average at the age of 80. For the age group of $35 - 44$, for instance, we vary ν from 0.0222 to 0.0278 at intervals of 0.0014. In this case, we have five values of ν. When matching up the stock investment with the SCF data, we take the median proportion of total wealth invested in stocks from age 35 through 44 (i.e., $\nu \in [0.0222, 0.0278]$) as the optimal investment proportion. When computing the optimal proportion of total wealth invested in stocks, we calibrate an initial endowment of financial wealth, w, and a constant stream of labor income per annum, $\epsilon = \xi I$, through the normalized cash-on-hand, i.e., the net worth normalized by income from the SCF.

Portfolio Share. The LNIS reduces the optimal portion of total wealth that should be invested in stocks (Table 3). The optimal investment strategy reported in the table differs from the wealth- and age- independent constant Merton (1969, 1971) investment rule. Our optimal investment strategy suggests that as people get older, their risky investment should be geared toward relatively safe assets; this advice is consistent with the rules of thumb proposed by financial advisers. The optimal risky portion itself is significantly $\leq$50%, as would be optimal in the Merton investment model. Our model generates empirically plausible values of 0 to 20% for optimal stock investment.

Interestingly, we show that people's risky investment ratio rises as their wealth increases, and this result is consistent with Wachter and Yogo (2010). The decision to invest in stocks is affected by two counteracting forces: a precautionary savings motive that decreases stock investment, and a diversification motive that increases stock investment. The presence of the LNIS itself increases undiversifiable background risk such as income risk, so precaution makes people conservative when taking on risk in the stock market. Accordingly, precautionary saving occurs. When saving occurs, the resources available for future investment are increased. Since the prices of risky investments are adjusted to increase their expected returns, the expected decline in labor income due to the LNIS can be partially offset. For risk diversification purposes, the share of total resources invested in the stock market would, thus, increase with an increase in wealth.

Human Capital Value. Neglecting the LNIS can be costly to individuals who aim to attain their consumption smoothing in terms of human capital aspect. We measure the value of human capital as the marginal rate of substitution between income and financial wealth. That is, the human capital value can be regarded as the individual's subjective marginal value of her income.

Similar to the relation of (22), the actuarial fair value (AFV) of future income discounted at the risk-free interest rate is[10]

$$AFV = \frac{r + \nu + \delta k}{(r + \nu + \delta)(r + \nu)}\epsilon,$$

which is the present value of income. We use AFV as a benchmark against the effects of the LNIS on the human capital value.

Definition 6.1. *Let $V(w, I; \delta)$ be the value function given in (3) with the Poisson intensity δ. Then, the value of human capital is defined as the marginal rate of substitution between income and financial*

[10] In the actuarial literature, it is common to use the risk-free interest rate for actuarial calculation purposes.

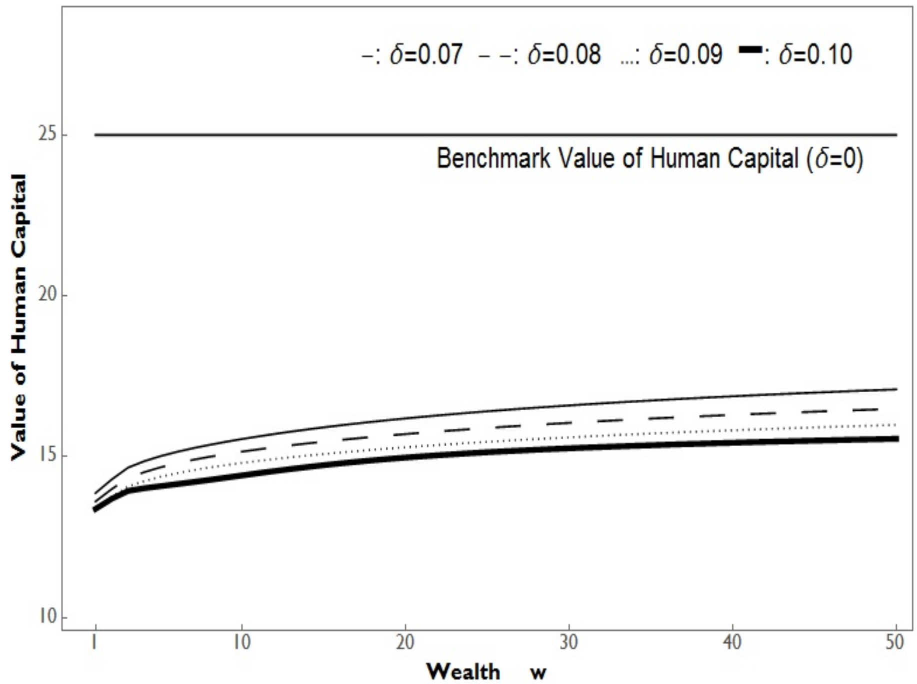

Figure 2: **Human capital value**. Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate). Note: The LNIS dramatically reduces the value of human capital, regardless of levels of financial wealth. The human capital value has an increasing and concave trend and the concavity strengthens as wealth decreases.

wealth, i.e.,

$$\frac{\partial V(w, I; \delta)}{\partial I} \Big/ \frac{\partial V(w, I; \delta)}{\partial w}.$$

The LNIS dramatically reduces the value of human capital, regardless of levels of financial wealth (Figure 2). As expected, the level of human capital decreases as the chance of income shock (δ) increases. The wage would thus decrease with the LNIS.

Interestingly, we could see an increasing and concave trend of the human capital value with respect to wealth, and the concavity strengthens as wealth decreases. This trend would be especially problematic for the poor people because they have very little residual income to save with the LNIS and therefore may be ill-prepared for the situations in its aftermath.

The increasing and concave human capital value in wealth would have relevance to the wealth concentration among the wealthy. While the rich theoretically do not save (Bewley, 1977; Campbell, 1987), they empirically have strong incentives to save and show the wealth concentration (Kaplan and

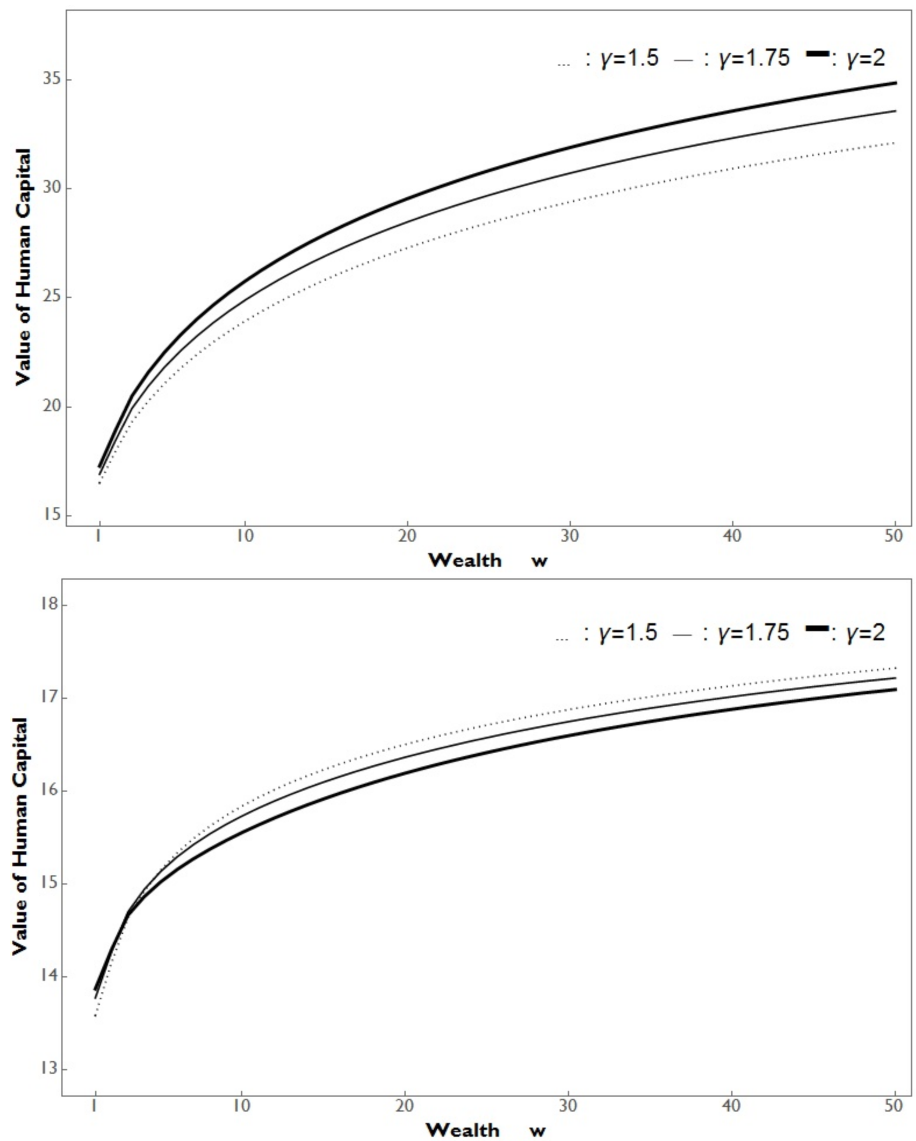

Figure 3: **The sensitivity of human capital value with respect to changes of risk aversion**. Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $\nu = 0.02$ (mortality rate), and $k = 0.2$ (recovery rate). Note: An increase of risk aversion increases the human capital value in the absence of the LNIS (top figure, $\delta = 0$). In contrast, in the presence of the LNIS, an increase of risk aversion decreases the human capital value (bottom figure, $\delta = 0.07$).

Rauh, 2013).[11] In terms of the human capital aspect, the top wealth-rich are high-income households and thus, their relatively high exposure of labor income to the LNIS is a likely contributor to their high precautionary savings rates for the preparation after the LNIS, which is consistent with the recent observations in the exposure of labor income to aggregate fluctuations (Parker and Vissing-Jrgensen, 2009; Guvenen *et al.*, 2017).

The effects of the LNIS outweigh the effects of risk aversion (Figure 3). An increase of risk aversion increases the human capital value in the absence of the LNIS. Intuitively, as an individual's risk aversion increases, she becomes increasingly likely to increase her investment in (seemingly) riskless human capital than in the risky assets (increasing her taking on market risk). In contrast, in the presence of the LNIS, an increase of risk aversion decreases the human capital value, so it no longer serves as a substitute of riskless assets, rather it resembles like a defaultable risky asset. Of course, the concern about the LNIS increases as risk aversion increases.

Utility Costs. We can measure utility costs of ignoring the LNIS as the wedge of value functions with and without the income shock. The costs can be thought of as the certainty equivalent wealth that is the greatest wealth the individual is willing to pay to reduce the probability or effect of the risk of catastrophic reduction in individual earnings.

Definition 6.2. *Let $\Delta(w)$ be the certainty equivalent wealth at initial wealth w, satisfying*

$$V(w - \Delta(w), I; \delta = 0) = V(w, I; \delta > 0),$$

where $V(w, I; \delta)$ is the value function given in (3) with the Poisson intensity δ.

Ignoring the LNIS can result in substantial utility costs in the form of certainty equivalent wealth (Figure 4). Obviously, the maximum payment that the individual should be willing to accept to eliminate the possibility of the catastrophic income shock (or reduce its negative effects) increases with the probability that the shock will occur.

[11] The wealthiest 400 Americans on the Forbes Magazine list own 1.5% of the total wealth in the US (Kaplan and Rauh, 2013).

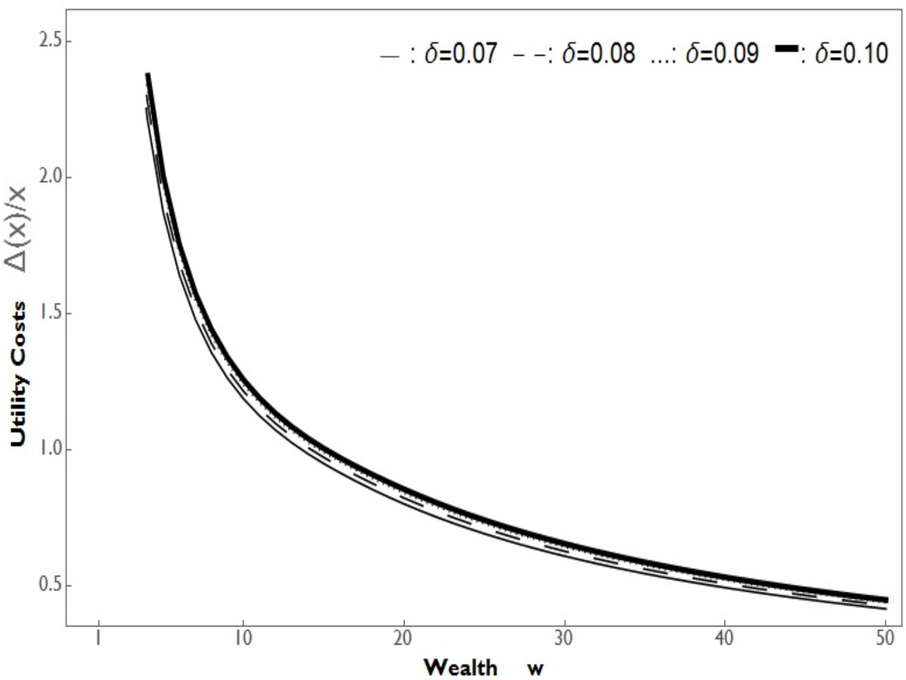

Figure 4: **Utility costs of ignoring a large, negative income shock**. Parameter Values: $\mu - r = 4\%$, $r = 2\%$, $\sigma = 20\%$, $\gamma = 2$, $\beta = 4\%$, $k = 20\%$, $L = 0$, $\epsilon = \xi I = 1$, and $\nu = 0.02$.

References

Bewley, T. 1977. The Permanent Income Hypothesis: A Theoretical Formulation. *Journal of Economic Theory.* **16**, 252–292.

Campbell, J. Y. 1987. Does Saving Anticipate Declining Labor Income? An Alternative Test of the Permanent Income Hypothesis. *Econometrica.* **55**, 1249–1273.

Dybvig, P. H., Liu, H. 2011. Verification Theorems for Models of Optimal Consumption and Investment with Retirement and Constrained Borrowing. *Mathematics of Operations Research.* **36** 620–635.

Friedman, M. 1957. A Theory of the Consumption Function. Princeton University Press, Princeton.

Guvenen, F., S. Schulhofer-Wohl, J. Song, and M. Yogo. 2017. Worker Betas: Five Facts about Systematic Earnings Risk. *American Economic Review.* **107**, 398–403.

Hall, R. E. 1978. Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence. *Journal of Political Economy.* **91**, 249–265.

Kaplan, S. N., J. D. Rauh. 2013. Family, Education, and Sources of Wealth among the Richest Americans, 1982-2012. *American Economic Review.* **103**, 158–162.

Karatzas, I., J. P. Lehoczky, S. E. Shreve, G. -L. Xu. 1991. Martingale and Duality Methods for Utility Maximization in an Incomplete Market. *SIAM Journal on Control and Optimization.* **29** 702–730.

Lucas, R. E. 1978. Asset Prices in an Exchange Economy. *Econometrica.* **46**, 1429–1445.

Merton, R. C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics.* **51**, 247–257.

Merton, R. C. 1971. Optimal Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory.* **3**, 373–413.

Modigliani, F., R. H. Brumberg. 1954. Utility Analysis and the Consumption Function: An Interpretation of Cross-Section data. Kenneth K. Kurihara, ed., Post-Keynesian Economics, New Brunswick, NJ. Rutgers University Press. 388–436.

Parker, J. A., A. Vissing-Jrgensen. 2009. Who Bears Aggregate Fluctuations and How? *American Economic Review.* **99**, 399–405.