Kidney Exchange with Immunosuppressants

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Abstract

Recent developments in immunosuppressive protocols have enabled patients to receive kidney transplants from biologically incompatible donors. We propose to use immunosuppressants as a part of kidney exchange program. We introduce the "pairwise cycles and chains (PCC)" solution and show that it is Pareto efficient, responsive, and maximizes the number of compatible transplants among all responsive solutions.

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1. Introduction

When a patient suffers from end-stage renal disease and has to receive a kidney transplant, several options are available depending on the immunological compatibility of the patient with her own donor. If the patient is compatible with her donor, a direct transplant within the patient-donor pair can be performed. Otherwise, the patient either registers on a waiting list to receive a transplant from a deceased donor or participates in a kidney exchange program where patients swap their donors to form compatible pairs (Roth et al. 2004). Unfortunately, these possibilities are limited given the growing number of patients waiting for transplants. However, recent developments in immunosuppressive protocols have introduced a new option: transplants from *incompatible* donors. This option, "desensitization", consists of the administration of several immunosuppressive medications (which we call "suppressants", for short) accompanied by a plasmapheresis treatment.[1] A patient going through desensitization becomes compatible with *any* donor, including her own.

Incompatible kidney transplants have been quite successful and shown to be a reliable alternative to other types of transplants or to dialysis.[2] Accordingly, the number of patients receiving incompatible transplants has been increasing in many countries. In South Korea, for example, the proportion of blood-type incompatible kidney transplants has increased from 4.7 percent to 23.1 percent of the total transplants from living donors during 2009-2016.[3] This option has largely replaced transplants through kidney exchanges, enabling patients to receive transplants directly from their own donors. Many other countries – such as Japan, France, and Sweden – are also using suppressants for transplants within pairs.

In the United States, in contrast, desensitization has not been used as much. So far, kidney exchange programs have allowed substantial welfare improvements, but there are still many patients who are waiting in the exchange pool to be matched or who cannot be matched with any donor due to particular biological traits. Obviously, these patients will benefit as the use of desensitization becomes more common, which will lower the medical cost of end-stage renal disease in the long run.

In this paper, we propose to incorporate desensitization into kidney exchange programs, as suppressants can be utilized more effectively when they are used to facilitate exchanges between pairs, rather than being only used for direct transplants within pairs. For this, pairs should be matched in coordination with the assignment of suppressants. To show how, we construct a new transplant system, which we call "pairwise cycles and chains (PCC)" solution. For each compatibility profile of participants, this solution selects an allocation of suppressants as well as a matching between patients and donors. We first explore the

[1] A quote from the New York Times article "*New Procedure Allows Kidney Transplants From Any Donor*" explains how this procedure works: "Desensitization involves first filtering the antibodies out of a patient's blood. The patient is then given an infusion of other antibodies to provide some protection while the immune system regenerates its own antibodies....[T]he person's regenerated antibodies are less likely to attack the new organ.... But if the person's regenerated natural antibodies are still a concern, the patient is treated with drugs that destroy any white blood cells that might make antibodies that would attack the new kidney. ...". This article was published on March 9, 2016 and is available at https://www.nytimes.com/2016/03/10/health/kidney-transplant-desensitization-immune-system.html.

[2] Immunosuppressants are currently being developed to eliminate *all* biological constraints including blood/tissue type incompatibility (Alexander et al. 1987; Gloor et al. 2003; Montgomery et al. (2011); Orandi et al. 2016). For the performance of these incompatible kidney transplants, see Takahashi et al. (2004), Tyden et al. (2007), Montgomery et al. (2012), Kong et al. (2013), Laging et al. (2014), Thielke et al. (2009), and Jin et al. (2012).

[3] The annual KONOS(Korean Network for Organ Sharing) reports are available at https://www.konos.go.kr/konosis/common/bizlogic.jsp.

implications of several welfare criteria that we formulate for this problem and then show that the PCC solution meets these criteria: it is *responsive*, *Pareto efficient*, and maximizes the number of compatible transplants – as well as the total number of transplants – among all *responsive* solutions.[4]

The literature on kidney exchange, stemming from the seminal work by Roth et al. (2004), provides rich analyses of centralized transplant systems under various institutional arrangements (Roth et al. 2005, 2007; Saidman et al. 2006; Sönmez and Ünver 2014). Most papers have taken compatibility profiles as a primitive, but these profiles can now be modified by utilizing newly developed transplant technologies. There are recent papers exploring these possibilities (Andersson and Kratz 2017; Sönmez et al. 2018) and our paper contributes to this line of research.[5]

The rest of this paper is organized as follows. Section 2 introduces the standard kidney exchange model without suppressants and defines the "priority-based" maximal matchings. Section 3 extends the model by introducing suppressants and establishes our main results. Section 4 contains a few concluding remarks.

2. Kidney Exchange without Immunosuppressants

We begin with the standard kidney exchange model without suppressants. There is a finite set N of patient-donor pairs. Let n be the number of pairs in N. Each pair $i \in N$ consists of patient i and donor i. A patient is either compatible or incompatible with a donor depending on immunological characteristics. We assume that each patient prefers a compatible transplant to an incompatible transplant, which she again prefers to no transplant.[6] Therefore, patient i's induced preference R_i over N is defined as follows: she prefers a pair whose donor is compatible with her to another pair whose donor is not; all pairs with compatible donors are equally desirable to her and so are all pairs with incompatible donors. Two pairs are *mutually compatible* if the patient of each pair is compatible with the donor of the other pair; similarly, two pairs are *mutually incompatible* if none of the patients is compatible with the donor of the other pair. For each $S \subseteq N$, let $R_S \equiv (R_i)_{i \in S}$ be the preference profile of pairs in S.

Keeping N fixed, a kidney exchange problem, or simply a *problem*, is defined as a preference profile $R = (R_i)_{i \in N}$.[7] We can also represent R as a graph as follows: (1) the nodes are pairs in N; (2) for each $i, j \in N$, if pairs i and j are mutually compatible, nodes i and j are connected by an undirected edge $(i - j)$; (3) for each $i, j \in N$, if patient i is compatible with donor j, but patient j is incompatible with donor i, nodes i and j are connected by a directed edge from j to i $(j \to i)$. Each pair is given a certain priority according to a linear ordering over N, which is determined by waiting time, age, and other health conditions (Roth et al. 2005). We denote this linear ordering by $\succ$ and write $i \succ j$ if patient i has a higher priority than patient j. Without loss of generality, let $1 \succ 2 \succ \cdots \succ n$.

[4] *Responsiveness* requires that all patients should become weakly better off as suppressants become available. For the formal definition, see Section 3.

[5] An independent work by Andersson and Kratz (2017) considers suppressants used in Sweden, which relax blood-type incompatibility, but not tissue-type incompatibility. Sönmez et al. (2018), on the other hand, consider a blood subtyping technology that enables transplants between certain incompatible blood-types.

[6] Note that in this section, no patient receives an incompatible transplant at any feasible matching. However, in Section 3, patients can receive incompatible transplants by using suppressants.

[7] Such dichotomous preferences have also been studied in a general matching context: Bogomolnaia and Moulin (2004) examine randomized solutions to achieve efficiency, fairness, and strategic requirements for pairwise matchings.

To accommodate physical and geographical restrictions in operating transplants, we assume that each kidney exchange takes place between two patient-donor pairs, involving four people at a time (we call this "pairwise exchange").[8] Formally, consider a mapping $\mu : N \to N \cup \emptyset$ such that for each $i, j \in N$, $\mu(i) = j$ if and only if $\mu(j) = i$. If $\mu(i) = j \in N$, pair i is matched to pair j and patient i receives a kidney from donor j; if $i \neq j$, pairs i and j form a *2-way match* at μ; if $i = j$, pair i *self-matches*. If $\mu(i) = \emptyset$, then pair i remains *unmatched*. Let $N(\mu) \equiv \{i \in N : \mu(i) \in N\}$ be the set of "matched" pairs at μ.

A mapping μ is a *matching* if (1) for each $i, j \in N$, $\mu(i) = j$ only if patient i is compatible with donor j, and (2) two distinct pairs $i, j \in N$ are not matched if patients i and j are compatible with their own donors.[9] Let $\mathcal{M}(R)$ be the set of matchings at R. Note that no patient is matched to an incompatible donor at any matching in $\mathcal{M}(R)$. Each patient is only concerned about her own match in comparing matchings. That is, she (weakly) prefers a matching to another if and only if she (weakly) prefers who she is matched with at the former to who she is matched to at the latter.

A matching is *Pareto efficient* at R if there is no other matching in $\mathcal{M}(R)$ that is weakly preferred by all patients and is strictly preferred by at least one patient. A matching μ is *maximal* at R if for each $\mu' \in \mathcal{M}(R)$, $|N(\mu')| \leq |N(\mu)|$. The structure of *maximal* matchings is fully characterized by the "Gallai-Edmonds decomposition" (Gallai 1963, 1964; Edmonds 1965; Bogomolnaia and Moulin, 2004; Roth et al. 2005). It is also well known that the set of maximal matchings coincides with the set of *Pareto efficient* matchings when each exchange takes place between two pairs. Let $\mathcal{M}^*(R)$ be the collection of maximal matchings at R.

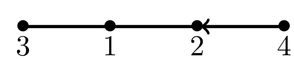
We now define "priority-based maximal matchings". Given $1 \succ 2 \succ \cdots \succ n$, let $\mathcal{M}_0 \equiv \mathcal{M}^*(R)$ and for each $k \in \{1, \cdots, n\}$,

$$\mathcal{M}_k \equiv \begin{cases} \{\mu \in \mathcal{M}_{k-1} : k \in N(\mu)\} & \text{if there is } \mu \in \mathcal{M}_{k-1} \text{ such that } k \in N(\mu); \\ \mathcal{M}_{k-1} & \text{otherwise,} \end{cases}$$

and lastly, let $\mathcal{M}^*_{\succ}(R) \equiv \mathcal{M}_n$. Note that $\mathcal{M}^*_{\succ}(R)$ is the subset of maximal matchings at which pairs with the highest possible priorities are matched (Roth et al. 2005; Nicolò and Rodríguez-Álvarez 2017). This set can be identified with linear programming or by using maximum weight matchings in a properly defined graph (Roth et al. 2005; Okumura 2014).

Example 1. Let R be a problem with 4 pairs such that

$$R = \left(\begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \\ \hline 2,3 & 1,4 & 1 & \emptyset \\ 1,4 & 2,3 & 2,3,4 & 1,2,3,4 \end{array} \right)$$



Note that pairs 1 and 3 are mutually compatible and so are pairs 1 and 2; patient 2 can receive a transplant from donor 4, but patient 4 cannot receive a transplant from any donor. For this profile, we have $\mathcal{M}^*(R) = \mathcal{M}_0 = \{\mu, \bar{\mu}\}$ where $\mu(1) = 3$, $\bar{\mu}(1) = 2$ and all other pairs remain unmatched. For the priority ordering $1 \succ 2 \succ 3 \succ 4$, we have $\mathcal{M}_0 = \mathcal{M}_1$ and $\mathcal{M}_2 = \mathcal{M}_3 = \mathcal{M}_4 = \mathcal{M}^*_{\succ}(R) = \{\bar{\mu}\}$. □

[8] The pairwise exchange models have been the main focus of the literature (Bogomolnaia and Moulin 2004; Roth et al. 2005, 2007; Saidman et al. 2006), but there are attempts to relax restrictions of the size of exchanges: for instance, see Ausubel and Morrill (2014).

[9] Condition (2) says that when two pairs are matched, at least one of them should not be able to make a self-match.

From the definition, the matchings in $\mathcal{M}^*_{\succ}(R)$ are equivalent in welfare: each pair is indifferent over all matchings in $\mathcal{M}^*_{\succ}(R)$. Therefore, the set of matched pairs remains the same across all $\mu^* \in \mathcal{M}^*_{\succ}(R)$. We choose any matching from $\mathcal{M}^*_{\succ}(R)$ and set it as a benchmark when we extend the model by introducing suppressants in Section 3.

3. Kidney Exchange with Immunosuppressants

We now introduce suppressants. Let R be a problem where no patient is compatible with her own donor. Let K be any non-negative integer representing the number of patients who can use suppressants for incompatible transplants. We consider all possible values of K and construct a solution that applies to any $K \in \mathbb{Z}_+ \cup \{0\}$.[10] If $K = 0$, for instance, we have a standard kidney exchange problem without suppressants; if $K \geq n$, any patient can receive an incompatible transplant, as in South Korea, in which case it is natural to minimize the use of suppressants.[11]

As explained in Introduction, if patient i uses a suppressant, she becomes compatible with all donors, including her own. For each $i \in N$, let $\hat{R}_i$ be the preference of patient i after receiving a suppressant. For each $S \subseteq N$ with $|S| \leq K$, let $R^S \equiv ((\hat{R}_i)_{i \in S}, R_{N \setminus S})$ and $\mathcal{M}(R^S)$ be the set of matchings at R^S such that each patient $i \in S$ is matched to an incompatible donor at R_i.[12] If patient i is matched to a compatible donor j at R_i, she receives a *compatible transplant* from donor j at μ. When patient i is matched to donor j who is incompatible at R_i but compatible at R^S_i, she receives an *incompatible transplant* from donor j. At each $\mu \in \mathcal{M}(R^S)$, let $C(\mu)$ be the set of pairs whose patients receive compatible transplants and $I(\mu)$ be the set of pairs whose patients receive incompatible transplants. Note that $|I(\mu)| \leq K$ should always hold and patient i receives an incompatible transplant if and only if $i \in S$.

We continue with the assumption that each patient prefers a compatible transplant to an incompatible transplant, which she again prefers to no transplant. We also assume that a patient is only concerned about her own match in comparing matchings. We define a **solution** as a pair (σ, φ) where for each R, (i) a **recipient choice rule** σ selects at most K pairs from N (namely, $\sigma(R) \subseteq N$ and $|\sigma(R)| \leq K$); and (ii) a **matching rule** φ, paired with σ, selects a set of matchings for $R^{\sigma(R)}$ (namely, $\varphi(R^{\sigma(R)}) \subseteq \mathcal{M}(R^{\sigma(R)})$). Let $\varphi^\sigma(R) \equiv \varphi(R^{\sigma(R)})$ be the set of matchings chosen by the solution.

3.1. Properties of Solutions

We propose several welfare criteria and explore their implications. Let (σ, φ) be a solution. Our first requirement says that all patients should become weakly better off as suppressants become available. For

[10] We note that K may not always be large due to several institutional constraints and a fixed time period under consideration. Incompatible transplants are operated by nephrologists and surgeons in specialized clinics. Given the limited personnel and the capacity of these clinics, including special equipments for desensitization – such as an automated centrifuge – the number of operations available for a certain period of time can be bounded. When these restrictions do not bind, we can set K to be sufficiently large.

[11] When the K-constraint binds, we solve a maximization problem of the number of compatible transplants (subject to *responsiveness*), which is a dual problem of minimizing the number of incompatible transplants; When the K-constraint does not bind, on the other hand, we solve a minimization problem. A solution to this maximization/minimization problem is the *Pairwise Cycles and Chains* solution presented in Section 3.2.

[12] Patients use suppressants only for incompatible transplants.

this, all patients who receive compatible transplants in the absence of suppressants (through exchanges) should receive *compatible* transplants at a matching selected by the solution. This implies that all patients chosen to use suppressants are those who cannot receive transplants in the absence of suppressants.

Responsiveness: For each R, each $\mu^* \in \mathcal{M}^*_{\succ}(R)$, and each $\mu \in \varphi^{\sigma}(R)$, $N(\mu^*) \subseteq C(\mu)$.

This is a fairness consideration that many practitioners in medical institutions may overlook. However, a social planner should make sure that no patient is penalized in the transition to a new system, so as to achieve minimal fairness in sharing the benefit of the new technology.[13] We are primarily interested in *responsive* solutions, but this requirement is trivially satisfied by several solutions: for example, consider a solution that disposes of all suppressants and chooses the benchmark matching. Consider another solution, the one that matches pairs through standard kidney exchange at first, removes all matched pairs, and then uses available suppressants for the remaining patients. Both solutions are trivially *responsive*, but they do not utilize suppressants effectively: the former obviously wastes them. The latter does not match as many patients as possible: for the profile in Example 1, for instance, suppose that $K = 1$. This solution matches pairs 1 and 2 at first, and then assigns a suppressant to either patient 3 or patient 4 for a self-match. Therefore, three patients are matched. However, there is an allocation that is better for everyone: assign a suppressant to patient 4 first, and then match pairs 1 and 3 and pairs 2 and 4, respectively.

To avoid the undesirable assignments described above, we next introduce two efficiency requirements on a solution. A standard efficiency requirement is that there should be no possible Pareto improvement from the choice made by the solution. Let R be a problem and $S, T \subseteq N$ be two recipient sets such that $|S|, |T| \leq K$. A matching $\mu \in \mathcal{M}(R^S)$ Pareto dominates another matching $\mu' \in \mathcal{M}(R^T)$ at R if all patients weakly prefer μ to μ' and at least one patient strictly prefers μ to μ' at R.

Pareto efficiency: For each R and each $\mu \in \varphi^{\sigma}(R)$, there are no $S \subseteq N$ with $|S| \leq K$ and $\bar{\mu} \in \mathcal{M}(R^S)$ such that $\bar{\mu}$ Pareto dominates μ at R.

Another efficiency requirement can be defined by means of the number of transplants: a solution should choose a set of recipients and matchings so as to maximize the total number of transplants as well as the number of compatible transplants.

Maximality: For each solution $(\bar{\sigma}, \bar{\varphi})$, each R, each $\mu \in \varphi^{\sigma}(R)$, and each $\bar{\mu} \in \bar{\varphi}^{\bar{\sigma}}(R)$, $|C(\mu)| \geq |C(\bar{\mu})|$ and $|N(\mu)| \geq |N(\bar{\mu})|$.

This requirement, a kind of "utilitarian efficiency", implies *Pareto efficiency*.[14] We next check if *maximality* is compatible with *responsiveness*. Unfortunately, we obtain an impossibility.

Proposition 1. No solution jointly satisfies *maximality* and *responsiveness*.[15]

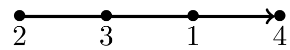
Proof. Let R be a problem with 4 pairs such that

[13]This requirement also provides patients a good incentive to participate in the new system: when patients are asked whether they stay in the existing system or participate in the new system, they choose the latter only if they become better off. *Responsiveness* guarantees a better outcome for every participant, enlarging the exchange pool and increasing gains from participation. Similar welfare requirements have also been studied in other contexts: for instance, see Kojima (2012) and Dŏgan (2016) for affirmative action in school choice problems and Thomson (2013) for various resource allocation problems.

[14]This logical relation is proven in Appendix A.

[15]This impossibility persists even if we modify *maximality* into a weaker version: For each solution $(\bar{\sigma}, \bar{\varphi})$, there are no R, $\mu \in \varphi^{\sigma}(R)$, and $\bar{\mu} \in \bar{\varphi}^{\bar{\sigma}}(R)$ such that $|C(\mu)| \leq |C(\bar{\mu})|$ and $|N(\mu)| \leq |N(\bar{\mu})|$ where at least one inequality holds strictly.

$$R = \left(\begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \\ \hline 3 & 3 & 1,2 & 1 \\ 1,2,4 & 1,2,4 & 3,4 & 2,3,4 \end{array} \right)$$



Let the priority ordering be $1 \succ 2 \succ 3 \succ 4$. In the absence of suppressants, there is one priority-based maximal matching μ^* such that $\mu^*(1) = 3$ and $\mu^*(2) = \mu^*(4) = \emptyset$. Suppose that $K = 1$. Any solution satisfying *maximality* should assign the suppressant to patient 1 and match pairs 1 and 4 and pairs 2 and 3, respectively. Note that patient 1 receives a compatible transplant from donor 3 at μ^*, but she receives an incompatible transplant from donor 4 when $K = 1$, becoming worse off. This violates *responsiveness.* $\square$

This result is disappointing, but recall that we are mainly interested in *responsive* solutions. We can define a weaker requirement than *maximality* by applying the same idea to *responsive* solutions only.

Constrained Maximality: For each *responsive* solution $(\bar{\sigma}, \bar{\varphi})$, each R, each $\mu \in \varphi^{\sigma}(R)$, and each $\bar{\mu} \in \bar{\varphi}^{\bar{\sigma}}(R)$, $|C(\mu)| \geq |C(\bar{\mu})|$ and $|N(\mu)| \geq |N(\bar{\mu})|$.

There is no logical relation between this requirement and *Pareto efficiency*; similarly, *Pareto efficiency* is logically independent of *responsiveness.*[16] Surprisingly, however, *Pareto efficiency* is implied by the combination of these two requirements:

Proposition 2. *Responsiveness* and *constrained maximality* jointly imply *Pareto efficiency.*

Proof. Let (σ, φ) be a solution satisfying *responsiveness* and *constrained maximality.* Suppose by contradiction that it is not *Pareto efficient.* Then, for some R and some $\mu \in \varphi^{\sigma}(R)$, there are $S \subseteq N$ with $|S| \leq K$ and $\bar{\mu} \in \mathcal{M}(R^S)$ such that $\bar{\mu}$ Pareto dominates μ at R. This implies that $C(\mu) \subseteq C(\bar{\mu})$ and $N(\mu) \subseteq N(\bar{\mu})$ with at least one proper inclusion relation. Since (σ, φ) is *responsive*, for each $\mu^* \in \mathcal{M}^*_{\succ}(R)$, we have $N(\mu^*) \subseteq C(\mu)$. From $C(\mu) \subseteq C(\bar{\mu})$, we obtain $N(\mu^*) \subseteq C(\bar{\mu})$. Now, consider another solution such that (1) for R, it chooses S and $\bar{\mu}$ and (2) for all other problems, it makes the same selection as (σ, φ). This solution is *responsive* by construction and at R, $|C(\mu)| \leq |C(\bar{\mu})|$ and $|N(\mu)| \leq |N(\bar{\mu})|$, with at least one strict inequality. This contradicts *constrained maximality* of (σ, φ). $\square$

3.2. Pairwise Cycles and Chains (PCC) Solution

We now ask whether *responsiveness* and *constrained maximality* are compatible. The answer is yes. To show this, we propose the "pairwise cycles and chains" solution, which is defined in several steps. Let K and R be given.

Pairwise cycles and chains (PCC) solution:

Step 1. Choose any matching from $\mathcal{M}^*_{\succ}(R)$ and denote it by μ^*.

Step 2. Define R^* as follows:

2.1 for each $i \in N(\mu^*)$ and each $j \in N$, if patient i is incompatible with donor j but patient j is compatible with donor i, let these pairs be mutually incompatible at R^*;

[16] Appendix A includes several examples showing these relations.

2.2 for each $i \notin N(\mu^*)$ and each $j \in N$, if patient i is incompatible with donor j, but patient j is compatible with donor i, let these pairs be mutually compatible at R^*;

2.3 for each $i \notin N(\mu^*)$, let patient i be compatible with her own donor i at R^*;

2.4 for each $i \in N$ and $j \in N$, if i and j are mutually compatible, let these pairs be mutually compatible at R^*; if i and j are mutually incompatible, let them be mutually incompatible at R^*.

Step 3. Identify $\left\{ \begin{array}{l} X \equiv \text{argmax}_{\mu \in \mathcal{M}(R^*): N(\mu^*) \subseteq N(\mu) \text{ and } |I(\mu)| \leq K} |N(\mu)| \\ \bar{X} \equiv \text{argmin}_{\mu \in X} |I(\mu)| \end{array} \right\}$ and choose $\bar{\mu} \in \bar{X}$.

Step 4. Let $\sigma(R) \equiv I(\bar{\mu})$ be the recipients of suppressant and $\{\mu \in \bar{X} : I(\mu) = I(\bar{\mu})\}$ be the set of matchings chosen by the PCC solution.

In Step 1, we identify a priority-based maximal matching μ^* in the absence of suppressants and set it as a benchmark matching. The PCC solution is designed so as to ensure that all patients matched at μ^* receive compatible transplants. Next, we modify R to R^* in Step 2, which is a key to achieving our requirements. In (2.1), we delete all directed edges from the pairs in $N(\mu^*)$ to the pairs outside $N(\mu^*)$. We thereby remove the possibility that any pair in $N(\mu^*)$ is matched to an incompatible donor. This guarantees *responsiveness* of the final matching. In (2.2), we change all the remaining directed edges into undirected edges; and in (2.3), we add self-directed edges for all nodes outside $N(\mu^*)$. In (2.4), we maintain all undirected edges as initially. It is (2.2)-(2.4) that allows us to find a largest set of self- and 2-way matches. In Step 3, we choose maximal matchings at R^*, subject to the number of patients receiving incompatible transplants not exceeding K and all patients in $N(\mu^*)$ being matched. If there are multiple matchings, we choose a matching that minimizes the use of suppressants.The choice is finalized in Step 4. This solution can be implemented by integer programming, as discussed in Appendix B.

Remark 1. A special case of this problem occurs when the K-constraint does not bind and all patients receive transplants (e.g., $K \geq n$). Formally, this is when there are $S \subseteq N$ with $|S| \leq K$ and $\mu \in \mathcal{M}(R^S)$ such that $S \cap N(\mu^*) = \emptyset$ and $|N(\mu)| = n$. For this case, the PCC solution chooses a "minimax" matching in Step 3, namely, a maximal matching that *minimizes the use of suppressants*, still subject to *responsiveness*. Therefore, all patients are matched and the number of compatible transplants is maximized.

Here is an instance of the PCC solution.

Example 2. Suppose that $K = 3$. Let R be a problem with 8 pairs such that

$$\left(\begin{array}{cccccccc} R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 \\ \hline 3,4 & 3 & 1,2 & \emptyset & 1,2 & \emptyset & \emptyset & 5,7 \\ N \setminus \{3,4\} & N \setminus \{3\} & N \setminus \{1,2\} & N & N \setminus \{1,2\} & N & N & N \setminus \{5,7\} \end{array} \right).$$

This problem is illustrated in Figure 1(i). Let the priority ordering be $1 \succ 2 \succ \cdots \succ 8$.

Step 1. We choose any matching from $\mathcal{M}^*_{\succ}(R)$. There is only one matching μ^* in $\mathcal{M}^*_{\succ}(R)$ such that $\mu^*(1) = 3$ and for each $i \notin \{1,3\}$, $\mu^*(i) = \emptyset$. Therefore, $N(\mu^*) = \{1,3\}$.

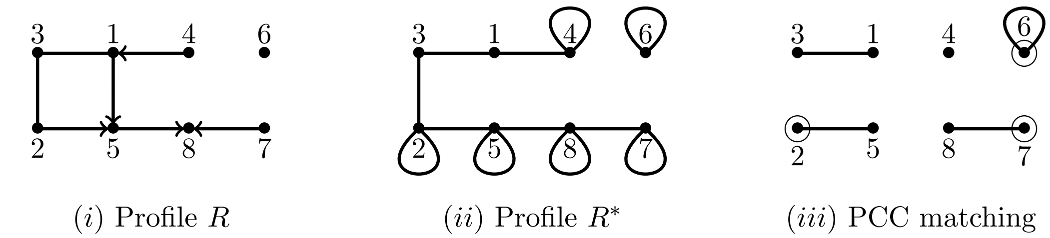


Figure 1: (i) and (ii): R^* is obtained from R, as directed in Step 2 of the PCC solution; (iii) All pairs except for pair 4 are matched at the final matching; the circled nodes represent the pairs who use suppressants.

Step 2. We derive R^* from R. Since $N(\mu^*) = \{1, 3\}$, we delete a directed edge from 1 to 5 and transform the following directed edges into undirected edges: edges from 4 to 1, from 2 to 5, from 5 to 8, from 7 to 8. We also add self-directed edges for each node other than 1 and 3. We represent new edges as dashed ones in Figure 1(ii).

$$\left(\begin{array}{cccccccc} R_1^* & R_2^* & R_3^* & R_4^* & R_5^* & R_6^* & R_7^* & R_8^* \\ \hline 3,4 & 2,3,5 & 1,2 & 1,4 & 2,5,8 & 6 & 7,8 & 5,7,8 \\ N\setminus\{3,4\} & N\setminus\{2,3,5\} & N\setminus\{1,2\} & N\setminus\{1,4\} & N\setminus\{2,5,8\} & N\setminus\{6\} & N\setminus\{7,8\} & N\setminus\{5,7,8\} \end{array}\right)$$

Step 3. We identify X and $\bar{X}$. We find that $X = \{\mu^{14}, \mu^{16}, \mu^{26}, \mu^{27}, \mu^{35}, \mu^{36}\}$ such that

$$\begin{pmatrix} \mu^{1i}(1) = 3 \\ \mu^{1i}(2) = 5 \\ \mu^{1i}(7) = 8 \\ \mu^{1i}(i) = i \end{pmatrix} \text{ with } i = 4 \text{ or } 6; \quad \begin{pmatrix} \mu^{2i}(1) = 4 \\ \mu^{2i}(2) = 3 \\ \mu^{2i}(5) = 8 \\ \mu^{2i}(i) = i \end{pmatrix} \text{ with } i = 6 \text{ or } 7; \quad \begin{pmatrix} \mu^{3i}(1) = 4 \\ \mu^{3i}(2) = 3 \\ \mu^{3i}(7) = 8 \\ \mu^{3i}(i) = i \end{pmatrix} \text{ with } i = 5 \text{ or } 6.$$

Since for all $\mu \in X$, $|I(\mu)| = 3 = K$, we have $\bar{X} = X$. We choose one of the matchings identified above.[17] For instance, let $\bar{\mu} \equiv \mu^{16}$.

Step 4. We finalize the choice. The PCC solution assigns suppressants to $\{2, 6, 7\}$ and choose the final matching μ^{16}, as illustrated in Figure 1(iii). $\square$

Now, here is our main result.

Theorem 1. The PCC solution satisfies *Pareto efficiency*, *responsiveness*, and *constrained maximality*.

By construction, the PCC solution is *responsive* and maximizes the total number of transplants subject to *responsiveness*. It is not obvious, however, that it also maximizes the number of *compatible* transplants among all *responsive* solutions. To prove this, we introduce a graph composed of two different types of edges and elaborate on it. Note again that we restrict our attention to *pairwise* exchanges only.

Proof. Responsiveness: Let R be a problem and $\mu^* \in \mathcal{M}^*_{\succ}(R)$ be a maximal matching chosen in Step 1 of the PCC solution. Let R^* be the profile derived from R in Step 2 of the PCC solution. Let (S, μ^{pcc}) be chosen by

[17] Step 3 of the PCC solution can be modified so as to use the priorities in selecting a matching from $\bar{X}$. The detail is deferred to Appendix C.

the PCC solution for R. We claim that each patient $i \in N(\mu^*)$ is never matched to an incompatible donor at any matching $\mu \in \mathcal{M}(R^*)$. Consider $i \in N(\mu^*)$ and any $j \in N$. If patient i is incompatible with donor j at R, then she is also incompatible with donor j at R^*. Therefore, it is not feasible to match patient i and donor j at any matching in $\mathcal{M}(R^*)$. On the other hand, $N(\mu^*) \subseteq N(\mu^{pcc})$ by the construction of X in Step 3 of the PCC solution. Altogether, we conclude that each $i \in N(\mu^*)$ is matched to a compatible donor at μ^{pcc}.

Constrained maximality: Let R be a problem and $(\sigma(R), \mu^{pcc})$ be a choice made by the PCC solution for R. Consider any *responsive* solution $(\bar{\sigma}, \bar{\varphi})$ and any $\mu \in \bar{\varphi}^{\bar{\sigma}}(R)$. We first show that the number of matched pairs at μ^{pcc} is at least as large as at μ.

Claim. $|N(\mu^{pcc})| \geq |N(\mu)|$.

Proof. We prove that $\mu \in \mathcal{M}(R^*)$. Consider any two distinct pairs, i and j. We show that if they can form a 2-way match at $R^{\bar{\sigma}(R)}$, they can also form a 2-way match at R^*. To form a 2-way match at $R^{\bar{\sigma}(R)}$, i and j should be mutually compatible at $R^{\bar{\sigma}(R)}$ and at least one of them should not belong to $\bar{\sigma}(R)$.[18]

Suppose that $i, j \notin \bar{\sigma}(R)$. For them to be mutually compatible at $R^{\bar{\sigma}(R)}$, they should be mutually compatible at R and therefore, they are mutually compatible at R^* trivially. Suppose that $i \in \bar{\sigma}(R)$ and $j \notin \bar{\sigma}(R)$. For them to be mutually compatible at $R^{\bar{\sigma}(R)}$, there should be a directed edge from i to j at R (equivalently, patient j should be compatible with donor i). By *responsiveness*, on the other hand, we should have $i \notin N(\mu^*)$. By the construction of R^*, i and j become mutually compatible at R^*. A symmetric argument applies to $i \notin \bar{\sigma}(R)$ and $j \in \bar{\sigma}(R)$. Therefore, if a 2-way match is formed at $R^{\bar{\sigma}(R)}$, then it can also be formed at R^*.

Next consider a single pair. We show that if it self-matches at $R^{\bar{\sigma}(R)}$, it can also self-match at R^*. This is because by *responsiveness*, $\bar{\sigma}(R) \cap N(\mu^*) = \emptyset$ and each pair in $N \setminus N(\mu^*)$ can self-match at R^* by (2.3). Altogether, $\mu \in \mathcal{M}(R^*)$. Since $\mu^{pcc} \in X$ and by the definition of X in Step 3 of the PCC solution, we conclude that $|N(\mu^{pcc})| \geq |N(\mu)|$. $\square$

By Claim, $|N(\mu^{pcc})| \geq |N(\mu)|$ and therefore, the proof is complete if we show that $|C(\mu^{pcc})| \geq |C(\mu)|$. Suppose by contradiction that $|C(\mu^{pcc})| < |C(\mu)|$. Since $|N(\mu^{pcc})| \geq |N(\mu)|$, we have $|I(\mu^{pcc})| > |I(\mu)|$.

We introduce a graph representing μ and μ^{pcc}. The nodes are the pairs in $N(\mu) \cup N(\mu^{pcc})$. We draw two types of edges as follows. Two nodes form an undirected solid edge if they make a 2-way match at μ^{pcc}; similarly, two nodes form an undirected dashed edge if they make a 2-way match at μ. By the definition of matchings, each node may involve at most one solid edge and at most one dashed edge (see Figure 2(ii) and Figure 3(iii)).

Consider a "sequence of alternating edges" in this graph, which is an ordered list of distinct edges between i_1 and i_2, i_2 and i_3, $\cdots$, and i_{k-1} and i_k, where $i_1 \neq i_2$ and a solid edge is followed by a dashed edge and a dashed edge is followed by a solid edge along the list. If $i_k = i_1$ for some $k \geq 3$, the sequence is an "alternating cycle". Since solid edges and dashed edges alternate along the alternating cycle, each node in the cycle should involve one dashed edge and one solid edge. On the other hand, if $i_k \neq i_1$ for some $k \geq 2$

[18] If both belong to $\bar{\sigma}(R)$, these two pairs do not form a 2-way match according to our definition of matchings.

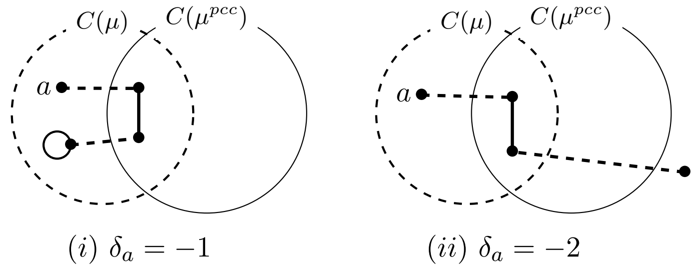


Figure 2: $|N_a \cap I(\mu)| + \delta_a < 0$ (Case 1) Choose an alternating path and then add self-directed edges to its nodes, if any. Among the nodes involving dashed edges, the nodes in $C(\mu)$ receive compatible transplants at μ; the nodes outside $C(\mu)$ receive incompatible transplants at μ.

and the sequence cannot be "extended further" – that is, i_1 forms exactly one edge with i_2 and i_k also forms exactly one edge with i_{k-1} – the sequence is an "alternating path" with two endnodes, i_1 and i_k.[19]

Let A be the set of all alternating cycles and alternating paths in the graph. For each $a \in A$, let N_a be the set of nodes that appear in a. Now, to each node in N_a, add a solid self-directed edge if it self-matches at μ^{pcc}; similarly, add a dashed self-directed edge if it self-matches at μ (see Figures 2(i), 3(i), and 3(ii)). If a is an alternating path, only the two endnodes may form self-directed edges, because all other nodes involve one solid edge and one dashed edge, according to the definition of alternating paths, forming 2-way matches at μ and μ^{pcc}. If a is an alternating cycle, no self-directed edge can be added. This is because each node of the alternating cycle involves one dashed edge and one solid edge and therefore, cannot self-match.

Consider a path or a cycle, $a \in A$ and the set of nodes appearing in a, N_a. Since the solid edges represent μ^{pcc},

- the number of pairs in N_a who are matched at μ^{pcc} (namely, $|N_a \cap N(\mu^{pcc})|$) is equal to the number of nodes in N_a involving solid edges in the graph. For instance, consider path a in Figure 2(i): three nodes involve solid edges of the path, and therefore, $|N_a \cap N(\mu^{pcc})| = 3$.

- Similarly, the number of pairs in N_a who receive compatible transplants at μ^{pcc} (namely, $|N_a \cap C(\mu^{pcc})|$) is equal to the number of nodes in $C(\mu^{pcc})$ involving solid edges in the graph; the number of pairs in N_a who receive incompatible transplants at μ^{pcc} (namely, $|N_a \cap I(\mu^{pcc})|$) is equal to the number of nodes outside $C(\mu^{pcc})$ involving solid edges. In Figure 2(i), for instance, $|N_a \cap C(\mu^{pcc})| = 2$ and $|N_a \cap I(\mu^{pcc})| = 1$.

More generally, for each $S \subseteq N$, $|S \cap N(\mu^{pcc})| = \sum_{i \in S} |\{i\} \cap N(\mu^{pcc})|$ and $|N(\mu^{pcc})| = \sum_{i \in N} |\{i\} \cap N(\mu^{pcc})|$. These equalities also hold if we replace $N(\mu^{pcc})$ with $I(\mu^{pcc})$ or $C(\mu^{pcc})$. For the other matching μ, we can similarly calculate $|N_a \cap N(\mu)|$, $|N_a \cap C(\mu)|$, and $|N_a \cap I(\mu)|$, by focusing on $C(\mu)$ and the dashed edges in the graph. For instance, in Figure 2(i), we have $|N_a \cap N(\mu)| = 4$, $|N_a \cap C(\mu)| = 2$, and $|N_a \cap I(\mu)| = 2$.

Let $\delta_a \equiv |N_a \cap N(\mu^{pcc})| - |N_a \cap N(\mu)|$. If δ_a is positive, μ^{pcc} matches δ_a more pairs than μ in N_a; if δ_a is negative, μ matches $|\delta_a|$ more pairs than μ^{pcc} in N_a. It is easy to check that $\delta_a \in \{-2, -1, 0, 1, 2\}$,

[19] By definition, both alternating cycles and alternating path are formed by at least two distinct nodes.

given all possible dashed/solid edges that the nodes in N_a may involve.[20] An alternating cycle, for instance, has $\delta_a = 0$ (the number of nodes involving the solid edges is the same as that involving the dashed edges); similarly, an alternating path starting and ending with the solid edges, without any self-directed edges, has $\delta_a = 2$ (the two endnodes involve solid edges, but not dashed edges).

Depending on the values of δ_a and $|N_a \cap I(\mu)|$, there are two possible cases of a.

Case 1. $|N_a \cap I(\mu)| + \delta_a < 0$. We show that a is an alternating path, not an alternating cycle, and should have $|N_a \cap I(\mu^{pcc})| = 0$, $|N_a \cap I(\mu)| = 1$, and $|N_a \cap C(\mu^{pcc})| + 1 = |N_a \cap C(\mu)|$. Note that for $|N_a \cap I(\mu)| + \delta_a < 0$ to hold, we should have $\delta_a \in \{-1, -2\}$, because $|N_a \cap I(\mu)| \geq 0$. If a were an alternating cycle, then $\delta_a = 0$. Therefore, a should be an alternating path.

Subcase 1.1. $\delta_a = -1$. For $|N_a \cap I(\mu)| + \delta_a < 0$ to hold, we should have $|N_a \cap I(\mu)| = 0$. By the definition of δ_a, on the other hand, $|N_a \cap N(\mu^{pcc})| = |N_a \cap N(\mu)| - 1$. That is, along the path, μ has no incompatible transplant, but has one more matched pair than μ^{pcc} (as in Figure 2(i)). Given these observations, we propose a new matching: choose the dashed edges of the nodes in N_a, while keeping the solid edges of all other pairs. This matching has one more matched pair than μ^{pcc}, but has at most as many incompatible transplants as μ^{pcc}, contradicting the choice of X in Step 3 of the PCC solution. Therefore, Subcase 1.1 is infeasible.

Subcase 1.2. $\delta_a = -2$. For $|N_a \cap I(\mu)| + \delta_a < 0$ to hold, $|N_a \cap I(\mu)| \leq 1$ (as in Figure 2(ii)). We show that $|N_a \cap I(\mu^{pcc})| < |N_a \cap I(\mu)|$. If not, $|N_a \cap I(\mu^{pcc})| \geq |N_a \cap I(\mu)|$. By the definition of δ_a, on the other hand, $|N_a \cap N(\mu^{pcc})| = |N_a \cap N(\mu)| - 2$. That is, μ has at most as many incompatible transplants as μ^{pcc}, but has two more matched pairs than μ^{pcc} along the path. Given these observations, we propose a new matching: choose the dashed edges of the nodes in N_a, while keeping the solid edges of all other pairs. This matching has two more matched pairs than μ^{pcc}, but has at most as many incompatible transplants as μ^{pcc}, contradicting the choice of X in Step 3 of the PCC solution. Altogether, $0 \leq |N_a \cap I(\mu^{pcc})| < |N_a \cap I(\mu)| \leq 1$, which implies that $|N_a \cap I(\mu^{pcc})| = 0$ and $|N_a \cap I(\mu)| = 1$. Altogether, we conclude that $|N_a \cap I(\mu^{pcc})| + 1 = |N_a \cap I(\mu)|$ and $|N_a \cap C(\mu^{pcc})| + 1 = |N_a \cap C(\mu)|$.

Case 2. $|N_a \cap I(\mu)| + \delta_a \geq 0$. We consider three subcases depending on the value of δ_a.

Subcase 2.1. $\delta_a \geq 0$. We show that $|N_a \cap I(\mu^{pcc})| \leq |N_a \cap I(\mu)| + \delta_a$. If not, $|N_a \cap I(\mu^{pcc})| > |N_a \cap I(\mu)| + \delta_a$ (as in Figure 3(i)). That is, the number of incompatible transplants at μ^{pcc} along the path is larger than that at μ by even more than δ_a. By the definition of δ_a, on the other hand, $|N_a \cap N(\mu^{pcc})| - |N_a \cap N(\mu)| = \delta_a$. That is, the number of matched pairs at μ^{pcc} along the path is larger than that at μ by δ_a. Given these observations, we propose a new matching: choose the dashed edges of the nodes in N_a (including self-directed dashed edges, if any) and add self-directed edges to the nodes in N_a involving no dashed edge (node i, for instance, in Figure 3(i)), while keeping the solid

[20] An alternating cycle has the same number of dashed edges and solid edges with no self-directed edges, so $\delta_a = 0$. Consider an alternating path starting in a solid edge and ending in a dashed edge. If both endnodes have self-directed edges, $\delta_a = 0$; similarly, if they have no self-directed edge, $\delta_a = 0$. If only one of them has a self-directed edge, then $\delta_a \in \{-1, 1\}$. Consider an alternating path starting and ending in solid edges. If none of the two endnodes have a self-directed edge, $\delta_a = 2$. If only one has it, $\delta_a = 1$ and if both have them, $\delta_a = 0$. A similar argument applies to an alternating path starting and ending in solid edges.

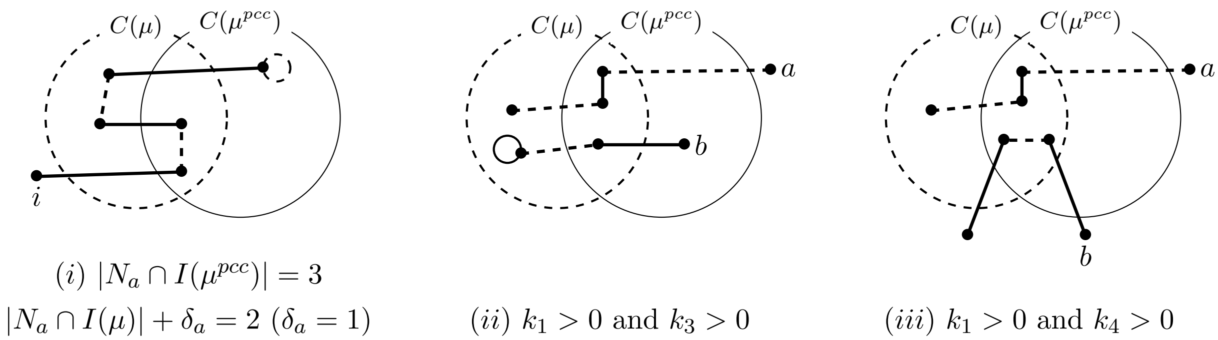


Figure 3: Proof of *constrained maximality.* (i) Choose the dashed edges and add a self-directed edge to i. Compared to the solid edges, this matching has the same number of transplants in total, but has one less incompatible transplant. (ii) Choose the dashed edges of paths a and b. Compared to the solid edges, this matching has one more transplant in total, but has the same number of incompatible transplants. (iii) Choose the dashed edges of the paths a and b. Compare to the solid edges, this matching has the same number of transplants in total, but has one less incompatible transplant.

edges of all other pairs. This matching has the same number of matched pairs as μ^{pcc}, but has a smaller number of incompatible transplants than μ^{pcc}, contradicting the choice of $\bar{X}$ in Step 3 of the PCC solution. Therefore, $|N_a \cap I(\mu^{pcc})| \leq |N_a \cap I(\mu)| + \delta_a$. Together with $|N_a \cap N(\mu^{pcc})| = |N_a \cap N(\mu)| + \delta_a$, we obtain $|N_a \cap C(\mu^{pcc})| \geq |N_a \cap C(\mu)|$.

Subcase 2.2. $\delta_a = -1$. We show that $|N_a \cap I(\mu^{pcc})| \leq |N_a \cap I(\mu)| - 1$. If not, $|N_a \cap I(\mu^{pcc})| > |N_a \cap I(\mu)| - 1$ (or equivalently, $|N_a \cap I(\mu^{pcc})| \geq |N_a \cap I(\mu)|$). By the definition of δ_a, on the other hand, $|N_a \cap N(\mu^{pcc})| - |N_a \cap N(\mu)| = -1$. That is, along the path, μ has one more matched pair than μ^{pcc}, but has at most as many incompatible transplants as μ^{pcc}. Given this observation, we propose a new matching: choose the dashed edges of the nodes in N_a (including self-directed dashed edges, if any), while keeping the solid edges of all other pairs. This matching has one more matched pair than μ^{pcc}, but has at most as many incompatible transplants as μ^{pcc}, contradicting the choice of X in Step 3 of the PCC solution. As in Subcase 2.1, we obtain $|N_a \cap C(\mu^{pcc})| \geq |N_a \cap C(\mu)|$.

Subcase 2.3. $\delta_a = -2$. We show that $|N_a \cap I(\mu^{pcc})| \leq |N_a \cap I(\mu)| - 1$. If not, $|N_a \cap I(\mu^{pcc})| > |N_a \cap I(\mu)| - 1$ (or equivalently, $|N_a \cap I(\mu^{pcc})| \geq |N_a \cap I(\mu)|$). Then, the similar argument of Subcase 2.2 applies and we can find a matching that has two more matched pairs than μ^{pcc}, but has at most as many incompatible transplants as μ^{pcc}, contradicting the choice of X in Step 3 of the PCC solution. Therefore, $|N_a \cap I(\mu^{pcc})| \leq |N_a \cap I(\mu)| - 1$. By the definition of δ_a, $|N_a \cap N(\mu^{pcc})| - |N_a \cap N(\mu)| = -2$ and therefore, $|N_a \cap C(\mu^{pcc})| \geq |N_a \cap C(\mu)| - 1$.

Cases 1 and 2 take care of all the pairs that appear in an alternating path or an alternating cycle in A. There may remain some pairs that are matched at μ or at μ^{pcc}, but do not appear in any alternating path or cycle. Let i be one of those pairs. Since pair i is not matched to any other pair in either matching, there are three possibilities:

(i) pair i self-matches both at μ and μ^{pcc} (namely, $|\{i\} \cap I(\mu^{pcc})| = |\{i\} \cap I(\mu)| = 1$);

(ii) pair i self-matches at μ but is unmatched at μ^{pcc} (namely, $|\{i\} \cap I(\mu^{pcc})| = 0 < |\{i\} \cap I(\mu)| = 1$);

(iii) pair i self-matches at μ^{pcc}, but is unmatched at μ (namely, $|\{i\} \cap I(\mu^{pcc})| = 1 > |\{i\} \cap I(\mu)| = 0$).

Note that $|\{i\} \cap C(\mu^{pcc})| = |\{i\} \cap C(\mu)| = 0$ in (i) to (iii). All the pairs that have not been discussed in Cases 1 and 2, (i) to (iii) are unmatched at μ^{pcc} and at μ.

Now, we return to the proof. Recall that $|C(\mu^{pcc})| < |C(\mu)|$ and $|I(\mu^{pcc})| > |I(\mu)|$.

(1) For the first inequality to hold, there should exist $S \subseteq N$ such that $|S \cap C(\mu^{pcc})| < |S \cap C(\mu)|$. From our analyses above, for this to hold, there should exist path a described in Case (1.2) or path a described in (2.3) such that $|N_a \cap C(\mu^{pcc})| = |N_a \cap C(\mu)| - 1$. In either case, $|N_a \cap N(\mu^{pcc})| = |N_a \cap N(\mu)| - 2$, $|N_a \cap I(\mu^{pcc})| = |N_a \cap I(\mu)| - 1$, $|N_a \cap C(\mu^{pcc})| = |N_a \cap C(\mu)| - 1$. Let k_1 be the number of alternating paths of this type.

(2) For the second inequality to hold, there should exist $S \subseteq N$ such that $|S \cap I(\mu^{pcc})| > |S \cap I(\mu)|$. From our analyses above, for this to hold, there should be either at least one pair i that self-matches at μ^{pcc} but is unmatched at μ, as described in (iii) above (let k_2 be the number of such pairs), or at least one alternating path $b \in A$ as discussed in Case 2.1, such that $\delta_b \in \{1, 2\}$ and $|N_b \cap I(\mu)| < |N_b \cap I(\mu^{pcc})| \leq |N_b \cap I(\mu)| + \delta_b$. For path b of this type, if any, there are three possibilities:

(2.1) $\delta_b = 1$ and $|N_b \cap I(\mu^{pcc})| = |N_b \cap I(\mu)| + 1$. Then $|N_b \cap C(\mu^{pcc})| = |N_b \cap C(\mu)|$. Let k_3 be the number of alternating paths of this type.

(2.2) $\delta_b = 2$ and $|N_b \cap I(\mu^{pcc})| = |N_b \cap I(\mu)| + 2$. Then $|N_b \cap C(\mu^{pcc})| = |N_b \cap C(\mu)|$. Let k_4 be the number of alternating paths of this type.

(2.3) $\delta_b = 2$ and $|N_b \cap I(\mu^{pcc})| = |N_b \cap I(\mu)| + 1$. Then $|N_b \cap C(\mu^{pcc})| = |N_b \cap C(\mu)| + 1$. Let k_5 be the number of alternating paths of this type.

Any $c \in A$ that is not covered in (1) or (2) has $|N_c \cap I(\mu^{pcc})| \leq |N_c \cap I(\mu)|$ and $|N_c \cap C(\mu^{pcc})| \geq |N_c \cap C(\mu)|$. Similarly, any self-matched pair j that has not been discussed in (1) and (2) above has $|\{j\} \cap I(\mu^{pcc})| \leq |\{j\} \cap I(\mu)|$ and $|\{j\} \cap C(\mu^{pcc})| = |\{j\} \cap C(\mu)|) = 0$.

Therefore, for $|C(\mu^{pcc})| < |C(\mu)|$ to hold, we should have $k_1 > k_5$. On the other hand, for $|I(\mu^{pcc})| > |I(\mu)|$ to hold, we should have $k_2 + k_3 + 2k_4 + k_5 > k_1$. Altogether, $k_1 > 0$ and $k_2 + k_3 + 2k_4 > 0$ should hold. That is, k_1 is positive and at least one of k_2, k_3, and k_4 is positive.

Since $k_1 > 0$, there is path a discussed in (1), along which μ has two more matched pairs than μ^{pcc}, but has only one more incompatible transplants than μ^{pcc}. If $k_2 > 0$, there is a pair i who self-matches at μ^{pcc}, receiving an incompatible transplant, but is unmatched at μ. We now propose a new matching: choose the dashed edges of the nodes in $N_a \cup \{i\}$, while keeping the solid edges of all other pairs. This matching has one more matched pair than μ^{pcc}, but has the same number of incompatible transplants as μ^{pcc}, contradicting the choice of X in Step 3 of the PCC solution. If $k_3 > 0$, there is path b discussed in (2.1), along which μ^{pcc} has one more matched pair than μ and one more incompatible transplant than μ (see Figure 3(ii)). We propose a new matching: choose the dashed edges over the nodes in $N_a \cup N_b$, while keeping the solid edges of all other pairs. This matching has one more matched pair than μ^{pcc}, but has the same number of incompatible transplants as μ^{pcc}, contradicting the choice of X in Step 3 of the PCC solution. Lastly, if $k_4 > 0$, there is path b discussed in (2.2), along which μ^{pcc} has two more matched pairs than μ and two more incompatible transplants than μ (see Figure 3(iii)). We propose a new matching: choose the dashed edges of the nodes in $N_a \cup N_b$, while keeping the solid edges of all other pairs. This matching has the same number of matched pairs as μ^{pcc}, but has one less incompatible transplants than μ^{pcc}, contracting the choice of $\bar{X}$ in Step 3 of the PCC solution. This completes the proof.

Pareto efficiency: By Proposition 2, this requirement is implied by *responsiveness* and *constrained maximality.* □

4. Concluding Remark

In this paper, we investigate the implications of introducing suppressants to the kidney exchange problem. We propose several welfare criteria in assigning suppressants and matching patients to donors. We introduce the PCC solution and show that it satisfies *Pareto efficiency*, *responsiveness*, and *constrained maximality.*

There remain several interesting issues. First, we may think of different procedures of assigning suppressants, instead of assigning K suppressants all at once as we do in this paper. For instance, suppose that we assign suppressants sequentially, one by one (or more generally, several units at a time). Each time, we apply the PCC solution to assign one unit of suppressant and let all patients who receive transplants leave the pool. It is easy to check that this sequential allocation of suppressants may make some patients worse off than under the all-at-once allocation of suppressants.

Second, we may consider adapting the deferred acceptance (DA) solution to our setting. Since there is a single priority ordering over patients, the DA solution can be defined as follows: Among all sets of 2-cycles and at most K number of 1- or 2-chains (such that each pair only appears in at most one of these 2-chains and 2-cycles), choose ones including the patient with the highest priority; among the resulting collections, choose ones including the patient with the second highest priority; and so on. From what we obtain, we choose the patients at the head of chains to be recipients of suppressant and let patients receive kidneys from donors along the directed edges in the cycles and chains. From this construction, we obtain a "stable" assignment: If a patient does not receive a transplant, then either (i) all patients with lower priorities do not receive transplants, or (ii) all available suppressants are assigned to patients with higher priorities. However, this solution violates *responsiveness*.

Third, there remains a participation issue, as a key feature of our proposal is that patient-donor pairs who are assigned suppressants still participate in kidney exchange pool, rather than opting out to make direct transplants within pairs. Provided that it does not make a significant difference from which donor a patient receives a kidney when using a suppressant, they would still participate to help other pairs receive transplants. Such an "altruistic" motivation is well documented in the standard kidney exchange program where *compatible* pairs participate (Sönmez and Ünver 2014; Roth et al. 2005; Gentry et al. 2007). There are other ways to provide an incentive to promote their participation more explicitly: for example, these patients could be provided with higher priorities when they need another operation in case of transplant failure. Sönmez et al. (2017) give insights into this possibility in a dynamic kidney transplant problem.

Appendix A

(1) *Maximality* implies *Pareto efficiency*: Suppose otherwise. Let (σ, φ) be a solution that satisfies *maximality* but not *Pareto efficiency*. Then, there are a problem R, a recipient set $S \subseteq N$ with $|S| \leq K$, and a matching $\bar{\mu} \in \mathcal{M}(R^S)$ such that $\bar{\mu}$ Pareto dominates μ at R for some $\mu \in \varphi^\sigma(R)$. Therefore, $C(\mu) \subseteq C(\bar{\mu})$ and $N(\mu) \subseteq N(\bar{\mu})$ with at least one proper inclusion, which implies that $|C(\mu)| \leq |C(\bar{\mu})|$ and $|N(\mu)| \leq |N(\bar{\mu})|$ with at least one strict inequality. Let $(\bar{\sigma}, \bar{\varphi})$ be the solution such that for R, $\bar{\sigma}(R) = S$ and $\bar{\varphi}^{\bar{\sigma}}(R) = \{\bar{\mu}\}$; for each other R, $(\bar{\sigma}, \bar{\varphi})$ makes the same selection as (σ, φ). This contradicts *maximality* of (σ, φ).

(2) *Pareto efficiency* and *constrained maximality* are logically independent: Let $K = 1$ and R be a problem with $N = \{1, 2, 3\}$. Suppose that patient 2 is compatible with donor 1 but all other patients are incompatible with each other donor. No patient can be matched in the absence of suppressants. Now, consider a solution that chooses $\{3\}$ as a recipient and chooses a matching composed of one self-match by pair 3 for this problem. This selection is *Pareto efficient*, but *constrained maximality* is violated at R, because if patient 1 were a recipient and pairs 1 and 2 were matched, the number of incompatible transplant remains the same, but the total number of transplant increases by 1. Conversely, *constrained maximality* does not imply *Pareto efficiency* either. Consider the problem in the proof of Proposition 1. Consider a solution that chooses $\{1\}$ as the recipient and chooses a matching μ composed of one 2-way match between pairs 2 and 3 and one self-match of pair 1. Obviously, this is not *Pareto efficient*. We show that this solution satisfies *constrained maximality*. Let $\bar{\mu}$ be any matching chosen by a *responsive* solution. Pair 1 should always be matched with pair 3 and either patient 2 or 4 can self-match at best by using a suppressant. Therefore, the number of compatible transplants is always 2 and the number of matched pairs will be 3 at most at $\bar{\mu}$, implying $|C(\bar{\mu})| \leq |C(\mu)|$ and $|N(\bar{\mu})| \leq |N(\mu)|$.

(3) *Pareto efficiency* and *responsiveness* are logically independent: Let (σ, φ) be the solution defined by setting for each R, $\sigma(R) = \emptyset$ and $\varphi^\sigma(R) = \mathcal{M}^*_\succ(R)$. It trivially satisfies *responsiveness*, but violates *Pareto efficiency*. Next, consider the problem R in the proof of Proposition 1. Consider the solution that chooses $\{1\}$ as the recipient and chooses a matching composed of 2-way matches between pairs 1 and 4 and pairs 2 and 3. The selection is *Pareto efficient*, but not *responsive*.

Appendix B

As a supplementary material, we discuss how to compute the PCC solution using integer programming as in Roth et al. (2004). Let R be a problem. We represent it as an $n \times n$ matrix $M = (m_{ij})_{i,j \in N}$ as follows. For each $i, j \in N$, if patient i is compatible with donor j, we write $m_{ij} = 1$; otherwise, $m_{ij} = 0$. Let $M^* \equiv (m^*_{ij})_{i,j \in N}$ be the similarly defined $n \times n$ matrix that represents R^* in Step 2 of the PCC solution.

Step 1 of the PCC solution: A priority-based maximal matching in $\mathcal{M}^*_\succ(R)$ is identified by a maximum weight matching of a properly defined graph (Okumura, 2014). For the details of this graph, see Okumura (2014). Let N^* be the set of pairs that are matched in this maximum weight matching, which corresponds to $N(\mu^*)$ in Step 1 of the PCC solution.

Step 3 of the PCC solution identifying $\boldsymbol{X}$: solve for the following integer programming problem.

$$\max_x \sum_{i,j\in N} x_{ij} \quad s.t. \quad \begin{array}{ll} (i) & x_{ij} \le m^*_{ij},\ x_{ij} = x_{ji} \in \{0,1\}, \quad \forall i,j \in N; \\ (ii) & \sum_{j\in N} x_{ij} \le 1,\ \forall i \in N; \\ (iii) & \sum_{j\in N} x_{ij} = 1,\ \forall i \in N^*; \\ (iv) & \sum_{i\in N: m_{ij}=0} x_{ij} \le K. \end{array}$$

Let x^* be a solution to this problem and let $k^* \equiv \sum_{i,j\in N} x^*_{ij}$.

Step 3 of the PCC solution identifying $\bar{\boldsymbol{X}}$: solve for the following integer programming problem.

$$\min_x \sum_{i,j\in N: m_{ij}=0} x_{ij} \quad s.t. \quad \begin{array}{ll} (i) & x_{ij} \le m^*_{ij},\ x_{ij} = x_{ji} \in \{0,1\}, \quad \forall i,j \in N; \\ (ii) & \sum_{j\in N} x_{ij} \le 1,\ \forall i \in N; \\ (iii) & \sum_{j\in N} x_{ij} = 1,\ \forall i \in N^*; \\ (iv) & \sum_{i\in N: m_{ij}=0} x_{ij} \le K; \\ (v) & \sum_{i\in N, j\in N} x_{ij} \ge k^*. \end{array}$$

Though integer programming with binary variables is NP-complete (Karp, 1972), there are various heuristic solutions to solve these problems that are used in practice. Several computation packages are also available (e.g., the mixed-integer linear programming tool at Matlab).

Appendix C

Theorem 1 says that *any* allocation chosen by the PCC solution satisfies *Pareto efficiency*, *responsiveness*, and *constrained maximality*. These requirements are still met even if we choose a particular PCC allocation by using the patients' priorities. Precisely, this is done by modifying the selection of $\bar{\mu}$ in Step 3 of the PCC solution: after defining X and $\bar{X}$ as in Section 3, we add the following:

Let $X_0 \equiv \bar{X}$ and for each $k \in \{1, \cdots, n\}$, let

$$X_k \equiv \begin{cases} \{\mu \in X_{k-1} : k \in N(\mu)\} & \text{if there is } \mu \in X_{k-1} \text{ such that } k \in N(\mu); \\ X_{k-1} & \text{otherwise,} \end{cases}$$

and let $\bar{\mu} \in X_n$.

For instance, consider the problem in Example 2. If we use the priority ordering $1 \succ 2 \succ \cdots \succ n$, we obtain $X_0 = X_1 = X_2 = X_3$, $X_4 = X_3 \setminus \{\mu^{16}\}$, $X_5 = X_4 \setminus \{\mu^{36}\}$, and $X_6 = X_7 = X_8 = \{\mu^{26}\}$. As for computation, it is enough to add the following iteration at the end of the integer programming presented in Appendix B.

3.1 Add a constraint of "$\sum_{j\in N} x_{1j} = 1$" to the existing constraints (i) to (v) of the minimization problem identifying $\bar{X}$. If there is a solution, keep this constraint with the existing constraints; otherwise, modify the constraint to $\sum_{j\in N} x_{1j} = 0$ and keep it with the existing constraints. Proceed to 3.2.

For each $k = \{2, \cdots, n\}$;

3.k Add "$\sum_{j\in N} x_{kj} = 1$" to the existing constraints of the minimization problem identifying $\bar{X}$. If there is a solution, keep this constraint with the existing constraints; otherwise, modify the constraint to $\sum_{j\in N} x_{kj} = 0$ and keep it with the existing constraints. Proceed to 3.$(k+1)$.

Let this process terminate at 3.n.

The resulting outcome is the priority-based matching that is chosen in Step 3 of the PCC solution.

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