Optimal Income Taxation and Disability Insurance for Households with Intra-Household Care*

Kyung-woo Lee†

Yonsei University

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Abstract

This paper characterizes the optimal income tax and disability insurance (DI) in the presence of formal and informal care for disabled people provided within households. In the model, each household is comprised of a disabled member and an able member who can provide "intra-household care" for the disabled member. Moreover, both the ability of able members and the level of disability of disabled members are heterogeneous across households. I first show that the intra-household care increases labor supply of disabled agents by reducing negative effects of disability on their labor supply. I also find that under reasonable assumptions optimal DI benefits are positive and progressive with respect to household earnings. Both features can be interpreted as redistribution mechanisms: positivity as the redistribution from the mildly disabled to the severely disabled, and progressivity as the redistribution from the more productive to the less productive.

*Very preliminary: Please do not cite or circulate.

†kwlee76@yonsei.ac.kr; School of Economics, Yonsei University, 50 Yonsei-ro, Seodaemun-gu, Seoul 03722, Korea

1 Introduction

Disability is one of the most serious risks over the life cycle. It can reduce individuals' earnings and welfare considerably due to the decline in labor productivity and limitations in various types of daily and economic activities. Moreover, disability has become more prevalent in OECD countries.[1] While individuals may insure themselves against the disability risk through their own savings, such self-insurance is often insufficient, especially in the case of early or persistent disability. For this reason, many countries have public disability insurance (DI) programs and a large volume of literature has been devoted to public DI programs' welfare and behavioral effects on disabled individuals.

In the literature, several papers recently turn their attention to the impact of disability on the families of disabled individuals.[2] Disability of an individual may affect the labor supply of his spouse or other members of the household in addition to his own labor supply. Moreover, disabled individuals often receive various types of care from their families or relatives, such as the assistance with activities of daily living (ADL) and medical care associated with the disability. Such care, which will be referred to as intra-household care throughout this paper, takes non-disabled household members time and monetary costs and hence may have an additional impact on their labor supply. Unfortunately, no paper in the literature has taken into account the household responses and intra-household care in the optimal design of DI, as standard papers in the literature simply assume single-person households in their models. Therefore, standard findings and policy proposals from those papers could be misleading to the extent that the household responses and intra-household care matter.

This paper contributes to the literature by addressing the issue. More specifically, this paper analyzes the optimal DI and income tax in the presence of household responses and intra-household care for disabled individuals. To this end, I consider a model with the features that account for the household effects on DI. First, each household is comprised of an able agent and a disabled agent, and both types of agents are heterogeneous across house-

[1]For the prevalence and welfare effects of disability for the U.S., see Meyer and Mok (2013), for example.

[2]See, for example, Fadlon and Nielsen (2015) and Autor et al. (2017), which are reviewed later in this section.

holds: able agents in labor productivity and disabled agents in the level of disability. Second, able agents can provide intra-household care for disabled agents to mitigate the labor disutility caused by disability. In this model, each household determines the labor supply of its disabled member (as well as its able member) by comparing the gain from the labor (wage) and the costs (utility costs of labor supply due to disability, costs of intra-household care, and forgone DI benefits). Hence, this model provides a useful framework to investigate the roles of behavioral responses of non-disabled individuals and intra-household care in the labor supply of disabled individuals and DI benefits.

Using the model, I first establish that intra-household care always increases the labor supply of disabled individuals. This outcome is intuitive because intra-household care would be provided only if the benefit from the care to a disabled agent, i.e., the reduction in the labor disutility, exceeds the cost of the care to an able agent. In other words, intra-household care takes place only if it can reduce the household's total cost of disability related to a disabled agent's labor supply. Consequently, intra-household care promotes the labor force participation of disabled individuals by reducing the costs of their labor supply for their households.

Then, I derive the optimal DI and income tax formulae that account for the household responses and intra-household care. In addition to usual behavioral effects of income taxation, the formulae feature the term that represents the effect of DI benefits on the labor supply of disabled agents. As the DI benefits increase, some households have disabled members withdraw from the labor force, which causes the tax revenue from the households to decline. This in turn reduces social welfare as the government has smaller tax revenue which could be used for welfare-improving redistribution. This effect is first incorporated to the optimal tax formulae by Kleven, Kreiner, and Saez (2009) in the context of income taxation of couples. This paper generalizes their analysis by explicitly accounting for the role of intra-household care as DI benefits affect the labor supply of disabled agents through the intra-household care channel.

Finally, I show that optimal DI benefits are always positive and decreasing in household earnings under reasonable assumptions. Both properties can be interpreted as redistribution from more fortunate households to less fortunate ones. More specifically, DI benefits should be positive to promote the redistribution from less severely disabled agents to more severely disabled agents. Likewise, DI benefits should decrease with household earnings to enhance the redistribution from more productive agents to less productive ones. *To test the validity of these results, I examine when the conditions for positive and progressive DI benefits are satisfied, assuming that functions associated with intra-household care exhibit constant absolute or relative risk aversion (CARA or CRRA). Such analyses reveal that they are indeed satisfied for realistic parameter values, which validates the characterization of the optimal DI benefits. [Needs calibration]*

This paper is related to several lines of literature. First, this paper builds on a large body of literature on optimal DI such as Diamond and Mirrlees (1978), Golosov and Tsyvinski (2006), and Low and Pistaferri (2015). While numerous papers analyzed optimal DI, no paper takes account of the household responses or intra-household care. Furthermore, most papers ignored the interaction between DI and income taxation, except for Lee (2015). The current paper makes a substantial contribution to the literature, as it is the first paper that characterizes the optimal DI and income tax with both household responses and intra-household care explicitly taken into account.

This paper is also associated with the literature on nonlinear optimal taxation following Mirrlees (1971) and Saez (2001).[3] Unlike the current paper, however, most papers in the literature analyzed optimal income taxation of individuals abstracting from the household responses. Kleven, Kreiner, and Saez (2009) recently addressed this problem by exploring the optimal income taxation of couples. Their setup is similar to that of the current paper in that both primary and secondary earners are heterogeneous in labor productivity and the cost of providing labor, respectively, and the distinction between two-earner and one-earner couples is the key to characterizing the optimal tax system. *The current paper generalizes their analysis by considering intra-household care and providing an in-depth analysis on the validity of key assumptions. [calibration should be added].*

The current paper is motivated by the findings in the literature on the impact of disabil-

[3]See Piketty and Saez (2013) for an excellent review.

ity or other health and income shocks to individuals on the labor supply of their spouses. In theory, individuals are likely to increase labor supply in response to such shocks to their spouses through the negative income effects. The increase in spousal labor supply was termed the added worker effect and empirically examined by many papers.[4] While some of the papers were specifically focused on the added worker effect on the disability or DI context, they did not incorporate their findings to the optimal design of DI. As discussed, one of the contributions of the current paper is to account for their findings in optimal DI and income tax.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 then characterizes the behavior of households such as labor supply of able and disabled agents and intra-household care, given a DI and income tax system. Based on the behavior of households, Section 4 derives the formulae for optimal DI and income tax and establishes the positivity and progressivity of optimal DI benefits. This analysis is followed by Section 5, which examines the validity of assumptions used to derive the key results in Section 4. Finally, Section 6 concludes.

2 Model

The economy is populated by a continuum of two-member households. Each household consists of two members: (i) the first member who is able, and (ii) the second member who is disabled. Each household is characterized with $n \in [\underline{n}, \bar{n}]$, the labor productivity of an able member, and $q \geq 0$, the severity of disability of a disabled member. Disability in this model is partial in the sense that disabled agents may still supply labor despite physical limitations. Both n and q differ across households, but they are assumed to be independent for analytic simplicity. The distributions of n and q are represented by distribution functions $F(n)$ and $P(q)$ and density functions $f(n)$ and $p(q)$. I assume that both n and q are only privately informed.

[4]For U.S., see Charles (1999), Cullen and Gruber (2000), Stephens (2002), Coile (2004), Blundell, Pistaferri, and Saporta-Eksten (2016), and Haan and Prowse (2017). For the analysis for other countries, refer to Fadlon and Nielsen (2015) for Denmark, Gallipoli and Turner (2011) for Canada, and Autor et al. (2017) for Norway.

In each household, an able member with n earns labor income z with labor hours z/n at the utility cost $h(z/n)$, which satisfies $h' > 0,\ h'' > 0$, and $h(0) = 0$. A disabled member may also provide labor supply l at a fixed wage rate $w > 0$. Following Kleven, Kreiner, and Saez (2009), I assume that l is binary, either 1 or 0. In addition to labor supply, an able member may also provide a disabled member in his household with various types of care such as medical care and assistance with activities of daily living. Such care will be referred to as *intra-household care* and denoted by $k \geq 0$. On the one hand, able agents incur a cost $v(k)$ to provide k units of intra-household care. $v(k)$ captures time costs of informal care and/or monetary costs of medical care and satisfies $v' > 0,\ v'' > 0$, and $v(0) = 0$. On the other hand, the intra-household care can improve the welfare of disabled agents. To capture this effect parsimoniously, I assume the labor disutility of a disabled agent is $ql[1 - m(k)]$. In this formulation, $m(k)$ represents the positive effects of intra-household care k and has the following properties: $m' > 0,\ m'' < 0,\ m(0) = 0$, and $\max_{k\geq 0} m(k) \leq 1$. A few points are worth further discussion. First, if a disabled agent forgoes labor supply, or $l = 0$, the disutility becomes zero and intra-household care becomes useless. Second, the labor disutility of disabled agents is proportional to the severity of their disability. Third, intra-household care can mitigate the labor disutility but only partially because $\max_{k\geq 0} m(k) \leq 1$.

Based on the discussion so far, I assume the utility function for a household with (n, q) as follows.

$$u(c, z, k, l) = c - h\left(\frac{z}{n}\right) - v(k) - ql[1 - m(k)], \tag{1}$$

where c is household consumption. The utility function is quasi-linear, which is standard in the literature, although it is augmented by the terms associated with intra-household care.[5] To describe the household budget constraint, notice first that income tax cannot be directly conditioned on n or q due to private information. Thus, I assume that income tax is based on (z, l), which fully characterizes household earnings because l is binary and w is common to all households. Hence, I denote by $T_l(z)$ the income tax for a household with an able member's earnings z and a disabled member's labor supply l. Using the income tax

[5] Following Kleven, Kreiner, and Saez (2009) and other papers in the literature, the concavity of utility function will be captured by the social welfare function, which I will describe in Section 4.

function, we can write the household budget constraint as

$$c = z + wl - T_l(z). \tag{2}$$

In the next section, I will describe the household problem and characterize the solution with the focus on the labor supply of disabled agents.

3 Optimal behavior of households

3.1 Household's problem

Each household chooses (c, z, k, l) to maximize the utility function (1) subject to the budget constraint (2). To solve this problem, I first characterize the households' behaviors conditional on l and then describe the choice of l. To this end, I define the value function of a household with (n, q) conditional on l as follows.

$$V_l(n, q) \equiv \max_{\{z,k\}} \left[z - T_l(z) - h\left(\frac{z}{n}\right) + wl - v(k) - ql\{1 - m(k)\} \right]. \tag{3}$$

The objective function in (3) is obtained from substituting (2) into (1). Let (z_l, k_l) denote the solution to the household problem (3) given l. Then, z_l should satisfy the following first- and second-order conditions (FOCs and SOCs, hereafter).

$$1 - T_l'(z_l) = \frac{1}{n} h'\left(\frac{z_l}{n}\right) \tag{4}$$

$$T_l''(z_l) + \frac{1}{n^2} h''\left(\frac{z_l}{n}\right) > 0 \tag{5}$$

There are implications of these conditions that will be used in the government's optimal taxation problem. First, z_l is decreasing in the marginal tax rate T_l'. To quantify this relationship, I denote by ε_l the elasticity of z_l with respect to the *marginal retention rate* $1 - T_l$, which is defined as

$$\varepsilon_l \equiv \frac{dz_l}{d\left(1 - T_l'\right)} \frac{1 - T_l'}{z_l} = \frac{h'\left(\frac{z_l}{n}\right)}{\frac{z_l}{n} h''\left(\frac{z_l}{n}\right)}. \tag{6}$$

Second, z_l increases with n, or more productive agents earn more. To see this, we take total differential to (4) to obtain

$$\frac{\partial z_l}{\partial n} = \frac{\frac{1}{n^2}\left(h' + \frac{z_l}{n} h''\right)}{T_l'' + \frac{1}{n^2} h''}, \tag{7}$$

which is positive because of $h' > 0,\ h'' > 0$, and (5). In this sense, $\partial z_l / \partial n > 0$ is equivalent to the second-order condition (5) given the assumptions on h.

As for k_l, note first that $k_0 = 0$ regardless of q because the marginal benefit from intra-household care is zero when $l = 0$. By contrast, k_1 can be positive and should satisfy the following FOC:[6]

$$qm'(k_1) \leq v'(k_1) \text{ with inequality iff } k_1 = 0. \tag{8}$$

Lemma 1 below summarizes the properties of k_1 implied by (8).

Lemma 1 *Define* $\chi \equiv \frac{v'(0)}{m'(0)},\ A_v(k) \equiv \frac{v''(k)}{v'(k)} > 0$, *and* $A_m(k) \equiv -\frac{m''(k)}{m'(k)} > 0$. *Then,* k_1 *and* $\partial k_1 / \partial q$ *have the following properties.*

1. *If* $q \leq \chi$, *then* $k_1 = 0$ *and* $\frac{\partial k_1}{\partial q} = 0$.

2. *If* $q > \chi$, *then* $k_1 > 0$ *and*

$$\frac{\partial k_1}{\partial q} = \frac{1}{q\left[A_v(k_1) + A_m(k_1)\right]} > 0. \tag{9}$$

Proof. Note that (8) can be rewritten as $q \leq \frac{v'(k)}{m'(k)}$ and $\frac{v'(k)}{m'(k)}$ is increasing in k. Thus, if $q \leq \chi = \frac{v'(0)}{m'(0)}$, the FOC holds as inequality, which means $k_1 = 0$ and $\partial k_1 / \partial q = 0$. However, if $q > \chi$, there is $k_1 > 0$ that satisfies (8) as equality. Also, if we take total differential to (8), we obtain (9). ■

According to Lemma 1, more severely disabled agents receive more intra-household care after a threshold level of disability $\chi = v'(0)/m'(0)$. Also, $\partial k_1 / \partial q$ increases as m becomes less concave and v becomes less convex, since A_v and A_m measure the curvature of v and m.

Another implication of the FOCs (4) and (8) is that z only depends on n whereas k only depends on q. Thus, the value function V_l can be decomposed as $V_l(n, q) = U_l(n) - C_l(q)$, where

$$U_l(n) \equiv z_l - T_l(z_l) - h\left(\frac{z_l}{n}\right) + wl, \tag{10}$$

$$C_l(q) \equiv v(k_l) + ql\left[1 - m(k_l)\right].$$

[6] Due to the assumptions on $h,\ m$, and v, the SOC for k_l is automatically satisfied.

$U_l(n)$ is interpreted as the utility associated with after-tax household earnings and labor disutility of an able agent. $C_l(q)$ represents the total cost of disability for a household because it includes both disutility due to disability and costs of intra-household care. Note that only $C_1(q)$ can be positive as $C_0(q) = 0$ due to $l = 0$ and $k_0 = 0$. Thus, I define

$$C(q) \equiv C_1(q) = v(k_1) + q[1 - m(k_1)] \tag{11}$$

and refer to $C(q)$ as the *disability cost of labor* for a household with (n, q).[7] As $C(q)$ is crucial to the decision making on disabled agents' labor supply, I summarize its key properties in the following lemma.

Lemma 2 *$C(q)$ is increasing and concave. It is partitioned into two parts as follows.*

1. *If $q \leq \chi \equiv \frac{v'(0)}{m'(0)}$, then $C(q) = q$, $C'(q) = 1$, and $C''(q) = 0$.*

2. *If $q > \chi$, then $C(q) < q$, $C'(q) = 1 - m(k_1) \in [0, 1)$, and*

$$C''(q) = -\frac{1}{q}\left[\frac{m'(k_1)}{A_v(k_1) + A_m(k_1)}\right] < 0.$$

Proof. If $q \leq \chi$, $k_1 = 0$ by Lemma 1. Then, $C(q) = q$ and $C'(q) = 1$. If $q > \chi$, $0 < m(k_1) \leq 1$ since $k_1 > 0$ by Lemma 1 and $m'(k) > 0$. This result implies $0 \leq C'(q) < 1$ because $C'(q) = 1 - m(k_1)$ by the envelope theorem. To prove $C(q) < q$,

$$C(q) = \int_0^{\chi} C'(x)\,dx + \int_{\chi}^{q} C'(x)\,dx < \int_0^{q} 1dx = q.$$

Finally, $C''(q) = -m'(k_1)\frac{dk_1}{dq}$. Using the formula for $\frac{dk_1}{dq}$ in Lemma 1, we obtain the equation for $C''(q)$ in Lemma 2. ■

All results of Lemma 2 are quite intuitive. Trivially, $C(q) = q$ when no intra-household care is provided to disabled agents. In the case of positive intra-household care, $C(q) < q$ because, if $C(q) \geq q$, zero intra-household care would be preferred to any positive intra-household care. Moreover, $C'(q) > 0$ indicates that an increase in q raises the disability cost of labor despite additional intra-household care. In other words, intra-household care

[7] "Labor" in the term obviously means the labor of the disabled member in a household. For the able member's labor, I will refer to $h(z/n)$ as "labor disutility" to avoid confusion.

cannot fully offset the negative effect of the increase in the severity of disability. Finally, $C''(q) < 0$ suggests that the total disability cost increases more slowly than q because of the endogenous intra-household care.

3.2 Labor supply of disabled agents

Now I turn to analyze the labor supply of disabled agents. A household with (n, q) chooses $l = 1$ if $V_1(n, q) \geq V_0(n, q)$, or equivalently, by $V_l(n, q) = U_l(n) - C_l(q)$, if

$$\Delta U(n) \equiv U_1(n) - U_0(n) \geq C(q). \tag{12}$$

In this condition, the left-hand side (LHS) captures the change in after-tax household income and labor disutility of an able member due to a disabled member's labor supply, whereas the right-hand side (RHS) represents the disability cost of labor. Thus, if (12) is satisfied for a household, the disabled member should supply labor because $\Delta U(n) \geq C(q)$ means that the labor supply of the disabled agent generates a positive net gain for the household. Notice for future reference that $C(q) = q$ if intra-household care were not available, because $k_1 = 0$ and $v(0) = 0$. Therefore, in this case, $\Delta U(n) \geq q$ replaces (12) as the condition for disabled agents' labor supply.

One of the key questions in this paper is how endogenous intra-household care affects the labor supply of disabled agents and corresponding responses of able agents. To address this question, let us define *labor thresholds* $q^*(n)$ with intra-household care and $q^*_{nc}(n)$ without intra-household care as follows.[8]

$$q^*_{nc}(n) = U_1(n) - U_0(n) = C(q^*(n)) \tag{13}$$

It is easy to see that this equation is based on (12). Since $C(q)$ is increasing in q by Lemma 2, $\Delta U(n) \geq C(q)$ if $q \leq q^*(n)$, whereas $\Delta U(n) < C(q)$ if $q > q^*(n)$. Then, by (12), a household with (n, q) chooses $l = 1$ if $q \leq q^*(n)$, and $l = 0$ if $q > q^*(n)$. In this model, therefore, mildly disabled agents with small q tend to supply labor $(l = 1)$, whereas severely disabled agents with large q tend to remain out of the labor force $(l = 0)$. For this reason,

[8] The subscript "nc" in q^*_{nc} stands for "no care."

we can define $B \equiv T_1 - T_0$ as DI benefits because it measures government transfers for households with $l = 0$, or equivalently, those with severe disability.

We can also formulate disabled agents' labor force participation rate based on the discussion so far. Let $\pi(n)$ and $\pi_{nc}(n)$ denote the participation rates of disabled agents in households with the same n, respectively, with and without intra-household care.[9] Formally, they are calculated as follows.

$$\pi(n) \equiv \Pr[\Delta U(n) \geq C(q)] = \Pr[q \leq q^*(n)] \tag{14}$$

$$\pi_{nc}(n) \equiv \Pr[\Delta U(n) \geq q] = \Pr[q \leq q^*_{nc}(n)]. \tag{15}$$

Moreover, using (14) and (15), we can calculate the economy-wide participation rate of disabled agents as

$$\begin{aligned} \pi &\equiv \int_{\underline{n}}^{\bar{n}} \pi(n) f(n)\, dn, \\ \pi_{nc} &\equiv \int_{\underline{n}}^{\bar{n}} \pi_{nc}(n) f(n)\, dn. \end{aligned}$$

As q^* and q^*_{nc} provide thresholds for the labor supply of disabled agents with and without intra-household care, we can examine the impact of intra-household care on the labor supply of disabled agents by comparing q^* and q^*_{nc}, as in the following proposition.

Proposition 1 *For any n, disabled agents' labor force participation rates and associated labor thresholds have the following properties.*

1. *If $\Delta U(n) \leq 0$, then $\pi_{nc}(n) = \pi(n) = 0$ and $q^*_{nc}(n) = q^*(n) = 0$.*
2. *If $0 < \Delta U(n) \leq \chi$, then $0 < \pi_{nc}(n) = \pi(n) < 1$ and $0 < q^*_{nc}(n) = q^*(n)$.*
3. *If $\Delta U(n) > \chi$, then $0 < \pi_{nc}(n) < \pi(n) \leq 1$ and $0 < q^*_{nc}(n) < q^*(n)$.*

Proof. In case 1, if $\Delta U(n) \leq 0$, no q can satisfy (12) because $C(q) \geq 0$ and $q \geq 0$. Hence result 1 follows.

[9] In this model, the labor participation rate is identical to the employment rate as there is no involuntary unemployment. Nonetheless, I will only use the labor participation rate because this model is focused on the supply side of labor.

In case 2, $0 < \Delta U(n) \leq \chi$ implies $0 < q^*_{nc}(n) \leq \chi$ by (13) and $C(q^*_{nc}(n)) = q^*_{nc}(n)$ by Lemma 2. Using (13) again, we obtain $q^*_{nc}(n) = q^*(n)$. Hence,

$$\pi(n) = \Pr[q \leq q^*(n)] = \Pr[q \leq q^*_{nc}(n)] = \pi_{nc}(n).$$

In case 3, $\Delta U(n) > \chi$ implies $q^*_{nc}(n) > \chi$. For such $q^*_{nc}(n)$, $C(q^*_{nc}(n)) < q^*_{nc}(n)$ by Lemma 2. By (13),

$$C(q^*_{nc}(n)) < q^*_{nc}(n) = \Delta U(n) = C(q^*(n)),$$

which implies $q^*_{nc}(n) < q^*(n)$ because $C(q)$ is increasing. Consequently,

$$\pi(n) = \Pr[q \leq q^*(n)] = \Pr[q \leq q^*_{nc}(n)] + \Pr[q^*_{nc}(n) < q \leq q^*(n)] > \pi_{nc}(n),$$

which concludes the proof. ■

Proposition 1 characterizes how intra-household care's effects on disabled agents' labor supply are associated with income tax. To better understand Proposition 1, I illustrate three cases of the proposition in Figure 1. In case 1, which corresponds to panel (a) of Figure 1, $\Delta U(n) \leq 0$. In this case, no positive q can satisfy (12), and, as a result, no disabled agents participate in the labor force. This result suggests that if the government were to induce some disabled agents to work, then it should design the income tax $T_0(z)$ and $T_1(z)$ in a way that $U_1(n) > U_0(n)$. In case 2, (12) can be satisfied for some positive q because $0 < \Delta U(n) \leq \chi$ and some disabled agents provide labor supply. As presented in panel (b) of Figure 1, however, we have $q^*_{nc}(n) = q^*(n) \leq \chi$ because $C(q) = q$ if $q \leq \chi$ by Lemma 2. Therefore, the income tax only incentivizes mildly disabled agents who do not receive any intra-household care, which makes intra-household care irrelevant in this case. By contrast, $\Delta U(n)$ is sufficiently large in case 3 and some disabled agents with $q > \chi$ also provide labor supply, as shown in panel (c) of Figure 1. Furthermore, since $q^*(n) > C(q^*(n)) = q^*_{nc}(n)$ by Lemma 2, intra-household care does promote the participation of disabled agents in labor force. In particular, while $l = 1$ for $q \leq q^*_{nc}(n)$ and $l = 0$ for $q > q^*(n)$ regardless of the availability of intra-household care, l changes from 0 to 1 for $q \in (q^*_{nc}(n), q^*(n)]$: they are induced to provide labor supply through the care by their household members, although they would remain out of the labor force without such care.

The results in Proposition 1 indicate that if sufficient incentives are provided to households with disabled agents through a proper design of $T_1(z)$ and $T_0(z)$, intra-household care can enhance disabled agents' labor force participation. This finding bears an interesting policy implication. While many countries try to encourage disabled people to work to reduce their fiscal burden, most policies are focused on the financial incentives for disabled people themselves. As such, the role of various types of care for disabled people within their households have been ignored. As shown in Proposition 1, however, if some policy can provide incentives for the intra-household care for disabled people, especially when they work, it can facilitate labor force participation of individuals with disability.

4 Optimal tax and DI benefits for households

In this section, I characterize the optimal tax and DI benefits in the model. I begin by describing the government's problem and derive the optimal tax formulae. Then, I analyze how optimal income tax depends upon the labor force participation of disabled agents.

4.1 Government's problem

In this economy, social welfare is defined as follows.

$$\int_{\underline{n}}^{\bar{n}} \int_{0}^{\infty} \Psi\left(V\left(n,q\right)\right) p\left(q\right) f\left(n\right) dq dn,$$

where Ψ is an increasing and concave function that translates household utility V to social welfare. It reflects the government's redistribution preferences and concavity of the household utility function. Based on the analysis in the previous section, social welfare can be rewritten as

$$\int_{\underline{n}}^{\bar{n}} \left[\int_{0}^{q^*(n)} \Psi\left(U_1\left(n\right) - C\left(q\right)\right) p\left(q\right) dq + \int_{q^*(n)}^{\infty} \Psi\left(U_0\left(n\right)\right) p\left(q\right) dq\right] f\left(n\right) dn. \tag{16}$$

The government chooses the tax policy (T_0, T_1) to maximize the social welfare (16) subject to the government budget constraint (GBC)

$$\int_{\underline{n}}^{\bar{n}} \left[T_1\left(z_1\right) P\left(q^*\left(n\right)\right) + T_0\left(z_0\right)\left\{1 - P\left(q^*\left(n\right)\right)\right\}\right] f\left(n\right) dn \geq 0, \tag{17}$$

and conditions on the households' choice of z_l and q^* : (13) for all $n \in [\underline{n}, \bar{n}]$, and

$$\dot{U}_l(n) = \frac{z_l}{n^2} h'\left(\frac{z_l}{n}\right), \text{ for any } n \in [\underline{n}, \bar{n}] \text{ and } l \in \{0, 1\}. \tag{18}$$

In the government's problem, the GBC (17) reflects the result that z_l is independent of q. Thus, $z_1(n)$ and $T_1(z_1(n))$ are applied to all households with given n and $q \leq q^*(n)$, and $z_0(n)$ and $T_0(z_0(n))$ to households with given n and $q > q^*(n)$. For this reason, $\int_0^{q^*} T_1(z_1)\, dP(q)$ and $\int_{q^*}^{\infty} T_0(z_0)\, dP(q)$ are simplified to $T_1(z_1) P(q^*(n))$ and $T_0(z_0) \{1 - P(q^*(n))\}$, respectively. Also in the problem, (18) is the envelope condition of the household problem with respect to n. Finally, it is noteworthy that the government's problem includes no condition on $C(q)$ or $k_1(q)$. This is due to the fact that the income tax $T_l(z)$ cannot influence the choice of k_l, as clear in (8) of the household problem. Hence, the government simply takes $C(q)$ and $k_1(q)$ as given in the government's problem.

4.2 Optimal tax formulae

To characterize the optimal tax formulae, I define the average social marginal welfare weight $g_l(n)$ as follows.

$$g_1(n) \equiv \frac{\int_0^{q^*(n)} \Psi'(U_1(n) - C(q))\, p(q)\, dq}{\lambda P(q^*(n))},$$

$$g_0(n) \equiv \frac{\int_{q^*(n)}^{\infty} \Psi'(U_0(n))\, p(q)\, dq}{\lambda [1 - P(q^*(n))]} = \frac{\Psi'(U_0(n))}{\lambda},$$

where λ is the Lagrangian multiplier for the GBC (17). As clear in the definition, $g_l(n)$ measures the average social value of a one-dollar increase in consumption for households with a given n and a choice of l. Notice that $g_l(n)$ is expressed in terms of dollars because of λ in the denominator. As we will see, their relative size will determine the direction of redistribution between households with $l = 1$ and those with $l = 0$.

To solve the government's problem, I make the following assumption for the SOC of the government's problem for z_l.

Assumption 1 *The function* $\Lambda(z) \equiv \frac{1 - \frac{1}{n} h'\left(\frac{z}{n}\right)}{\frac{1}{n}\left[h'\left(\frac{z}{n}\right) + \frac{z}{n} h''\left(\frac{z}{n}\right)\right]}$ *is decreasing in* z.

Assumption 1 is satisfied, for example, for standard functions such as $h(z/n) = (z/n)^{1+1/\varepsilon} / (1+1/\varepsilon)$ with $\varepsilon > 0$. Given this assumption, we can solve the government's problem to derive the optimal tax formulae, which are presented in the following proposition.

Proposition 2 *Under Assumption 1 and if there is no bunching, the optimal tax rates satisfy the following formulae for any* n.

$$\frac{T_1'}{1-T_1'} = \frac{1+\varepsilon_1}{\varepsilon_1} \frac{1}{P(q^*) f(n) n} \int_n^{\bar{n}} \left[(1-g_1) P(q^*) - \frac{T_1 - T_0}{C'(q^*)} p(q^*) \right] f(x)\, dx, \tag{19}$$

$$\frac{T_0'}{1-T_0'} = \frac{1+\varepsilon_0}{\varepsilon_0} \frac{1}{[1-P(q^*)] f(n) n} \int_n^{\bar{n}} \left[(1-g_0)[1-P(q^*)] + \frac{T_1 - T_0}{C'(q^*)} p(q^*) \right] f(x)\, dx, \tag{20}$$

along with the SOC with respect to $q^*(n)$:

$$(T_1 - T_0) \left[\frac{p'(q^*)}{p(q^*)} - \frac{C''(q^*)}{C'(q^*)} \right] < 0. \tag{21}$$

In the above conditions, all terms outside the integrals are evaluated at n *and the terms inside the integrals at* x.

Proof. See Appendix A.1. ■

Proposition 2 presents the optimal formulae for marginal tax rates and the SOC of the government's problem. The optimal tax formulae (19) and (20) are distinct from the corresponding formulae in Kleven, Kreiner, and Saez (2009) due to the presence of the marginal disability cost of labor $C'(q^*)$. To better understand the role of this novel term, let us examine (19) in detail, as (20) can be explained similarly.

To this end, suppose that (T_1, T_0) is the optimal income tax system. Then, no perturbation could change social welfare because, otherwise, some reform could improve social welfare. Based on this principle, I examine the optimality of $T_1(z_1(n))$ for households with n by analyzing the effects of raising T_1' for earnings $z_1(n')$ of households with $n' \in [n - dn, n]$ by a constant $d\tau$. By construction, such a perturbation increases $T_1(z_1(x))$ for $x \geq n$ by a constant $dT \equiv d\tau \times dz_1$, where $dz_1 = z_1(n) - z_1(n - dn)$. Now let us investigate the welfare effects of the perturbation, which will be expressed in terms of government's tax revenue used for redistribution.

First, social welfare changes due to the mechanical increase in tax payments by households with $x \geq n$ and $l = 1$. On the one hand, the additional tax revenue $dT \int_n^{\bar{n}} P(q^*) f(x) dx$ can be redistributed to improve social welfare. On the other hand, as each of such households reduces consumption by the additional tax payment dT, social welfare falls by $dT \int_n^{\bar{n}} g_1 P(q^*) f(x) dx$. Thus, the combined effect on social welfare is calculated as

$$dW_1 = dT \int_n^{\bar{n}} \left[(1 - g_1) P(q^*) \right] f(x) dx.$$

Second, social welfare also falls because of the behavioral responses of the two-earner households with $n' \in [n - dn, n]$, who face higher marginal tax rates. This welfare effect is represented by the fall in tax revenue from such households. To measure this effect, let $\delta z_1(n')$ denote the fall in $z_1(n')$ for households with $n' \in [n - dn, n]$ due to the rise in T_1' by $d\tau$. Then, their tax payment falls by $T_1'(z_1(n')) \delta z_1(n') P(q^*(n')) f(n')$, which is integrated over $[n - dn, n]$ to yield the total tax reduction:

$$\begin{aligned} dW_2 &= \int_{n-dn}^{n} T_1'\left(z_1\left(n'\right)\right) \delta z_1\left(n'\right) P\left(q^*\left(n'\right)\right) f\left(n'\right) dn' \\ &\approx T_1'(z_1(n)) \delta z_1(n) P(q^*(n)) f(n) dn. \end{aligned}$$

Using the total differential of (4), I obtain

$$\delta z_1(n) \left[T'' + \frac{1}{n^2} h'' \left(\frac{z_1}{n} \right) \right] = -d\tau.$$

Combining this condition with (7), I can rewrite $\delta z_1(n)$ as

$$\delta z_1(n) = -d\tau \frac{1}{T'' + \frac{1}{n^2} h''} = -d\tau \frac{\partial z_1}{\partial n} \frac{1}{\frac{1}{n^2}\left(h' + \frac{z_1}{n} h''\right)}.$$

Substituting this into the equation for dW_2 yields

$$dW_2 = -T_1' \frac{\partial z_1}{\partial n} \frac{1}{\frac{1}{n^2}\left(h' + \frac{z_1}{n} h''\right)} P(q^*) f(n) d\tau dn = -T_1' \frac{1/h'}{\frac{1}{n^2}\left(h' + \frac{z_1}{n} h''\right)/h'} P(q^*(n)) f(n) d\tau dz_1$$

which can be rewritten as

$$dW_2 = -\frac{T_1'}{1 - T_1'} \frac{\varepsilon_1}{1 + \varepsilon_1} n P(q^*(n)) f(n) dT,$$

due to (4), (6), and $d\tau dz_1 = dT$.

In the case of individual income taxation, these two effects would be sufficient as all other effects are cancelled out due to the envelope theorem. In this model, however, the tax perturbation has an additional welfare effect, which is associated with the labor supply decision of disabled agents. For disabled agents in households with $x \geq n$, the increase in T_1 reduces $\Delta U(x)$ by dT, which in turn lowers $q^*(x)$ by $dq^*(x)$ in a way that $C(q^*(x)) - C(q^*(x) - dq^*(x)) = dT$. The approximation of this equation by differentials yields

$$dq^*(x) = \frac{dT}{C'(q^*(x))}.$$

The change in $q^*(x)$ implies that some disabled agents exit from the labor market by choosing $l = 0$. The number of such "switchers" in households with (x, q) is measured as

$$[P(q^*(x)) - P(q^*(x) - dq^*(x))] f(x) \approx p(q^*(x)) dq^*(x) f(x),$$

where I use approximation by differentials again. Then, as tax revenue from each of such households falls by $T_1 - T_0$, total change in tax revenue due to the change in l is calculated as

$$dW_3 = -dT \int_n^{\bar{n}} \frac{[T_1 - T_0]}{C'(q^*)} p(q^*) f(x) dx.$$

For the optimality of the pre-perturbation income tax system, $dW_1 + dW_2 + dW_3 = 0$, which implies

$$\int_n^{\bar{n}} [(1 - g_1) P(q^*)] f(x) dx - \frac{\varepsilon_1}{1 + \varepsilon_1} \frac{T_1'}{1 - T_1'} n f(n) P(q^*) - \int_n^{\bar{n}} \frac{[T_1 - T_0]}{C'(q^*)} p(q^*) f(x) dx = 0.$$

It is straightforward to see that this condition is equivalent to the optimal tax formula (19) for households with $l = 1$ for disabled members.

In the optimal tax formulae, the $1/C'(q^*(n))$ term appears in both (19) and (20), but with the opposite signs. This is because it represents the impact of a change in T_1 or T_0 on disabled agents' labor force participation. Intuitively, if $T_1(z_1(n))$ rises or if $T_0(z_0(n))$ falls by a dollar with all other things left unaffected, then the utility gap between a two-earner household and a one-earner household, $\Delta U(n)$, is reduced by a dollar. Hence, the labor supply becomes less attractive to disabled agents, as reflected by a corresponding one-dollar decline in $C(q^*(n))$ by (13). As a result, the labor force participation rate of disabled agents falls because $q^*(n)$ decreases by $dq^*/dC = 1/C'(q^*(n))$. This analysis clearly shows that $1/C'(q^*(n))$ quantifies the reduction in $q^*(n)$ in response to a one-dollar increase in $T_1(z_1(n))$, or a one-dollar decrease in $T_0(z_0(n))$.

4.3 Optimal design of DI benefits

As noted, DI benefits can be expressed as $B(n) = T_1(z_1(n)) - T_0(z_0(n))$ in this model because it is the government's transfer to households with $l = 0$ or relatively seriously disabled agents. Thus, we should understand the properties of T_1 and T_0 to characterize the optimal DI benefits $B(n)$. To this end, we could use the optimal tax formulae (19) and (20) in Proposition 2. Those formulae, however, are not so helpful to understand $B(n)$ because they do not tell which of T_0' and T_1' is larger. Therefore, it is generally ambiguous whether $B(n)$ is positive and how $B(n)$ changes with n, according to Proposition 2.

From the social insurance perspective, however, one could argue that DI benefits should be positive for all n and decreasing in n. Since both the ability of able agents and the severity of disability are random in this model, we can interpret the households with low n or high q as less fortunate and those with high n or low q as more fortunate. For redistribution to improve social welfare, therefore, $B(n) > 0$ can be desirable because $B(n)$ represents the redistribution from households with low q to those with high q. Similarly, $B'(n) < 0$ can be also justified because the condition means that households with high n receive less DI benefits than those with low n.

Based on the discussion so far, I explore conditions under which $B(n) > 0$ and $B'(n) \leq 0$ for any n. It turns out that $B(n)$ has those properties under assumptions listed below. In the next section, I examine their validity and show that they tend to be satisfied for reasonable parameter values if functions associated with intra-household care, such as $v(k)$, $m(k)$, and $P(q)$, are assume to exhibit CARA or CRRA. Now I present the assumptions required for $B(n) > 0$ and $B'(n) \leq 0$ and explain them.

Assumption 2 $\Psi'(U - C(q))$ *is convex in* q.

Assumption 3 *For all* q, $qp(q)/P(q) \leq 1$.

Assumption 4 *A function* $M(q) \equiv \frac{w-C(q)}{C'(q)} \frac{p(q)}{P(q)(1-P(q))}$ *is decreasing in* q.

Assumption 2 is needed for the society to have sufficiently strong preferences for redistribution. Taking the second derivative of $\Psi'(U - C(q))$, we can see Assumption 2 implies

$$\frac{d^2 \left[\Psi'(U - C(q))\right]}{dq^2} = \Psi'''(U - C(q)) \left\{C'(q)\right\}^2 - \Psi''(U - C(q)) C''(q) \geq 0.$$

Because $\Psi'' < 0$ and $C'' \leq 0$, Assumption 2 requires that Ψ''' should be positive and sufficiently large. For example, if we use $\Psi(V) = V^{1-\theta}/(1-\theta)$ with $\theta > 0$, the assumption means that θ should be large enough. However, the analysis in the next section reveals that it is satisfied even for a reasonably small θ. Assumptions 3 and 4 are technical assumptions. Assumption 3 is satisfied for $P(q) = (q/q_{\max})^{\eta}$ if $0 < \eta \leq 1$, or for $P(q) = 1 - \exp(-\eta q)$ with $\eta > 0$. Notice that for both functions, $p(q)$ is decreasing in q, which means that as disability becomes severe, the number of agents declines. This property seems reasonable for a function to match the distribution of the severity of disability. Finally, the validity of Assumption 4 depends mostly on the behavior of $P(q)$. If q is unbounded as in the exponential distribution $P(q) = 1 - \exp(-\eta q)$, $\eta > 0$, the assumption tends to be satisfied. By contrast, if q is bounded by $q_{\max}$ so that $P(q_{\max}) = 1$, then $\frac{1}{1-P(q)}$ eventually explodes to the infinity. Thus, the expression in the assumption could eventually be increasing. As will be discussed later, however, we can avoid this issue by imposing a reasonable condition. With the assumptions discussed so far, I now present the proposition that characterizes $B(n)$.

Proposition 3 *Under Assumptions 1-4, and assuming no bunching at the optimum, the optimal DI benefits are characterized as follows.*

1. $B'(n) \leq 0$, *or equivalently,* $T_1' \leq T_0'$ *for all* n.

2. $B(n) \geq B(\bar{n}) > 0$ *for all* $n \leq \bar{n} < \infty$.

3. *Disabled agents' labor force participation rate increases with* n, *that is,* $dq^*/dn \geq 0$.

4. *Define* $\tau(n)$ *as the labor income tax rate for a disabled agent in a household with* n :

$$\tau(n) \equiv \frac{w - \Delta U(n)}{w} = \frac{w - C(q^*(n))}{w}. \tag{22}$$

Then, $d\tau/dn \leq 0$.

Proof. See Appendix A.2. ■

First of all, Proposition 3 establishes two important results: $B'(n) \leq 0$ and $B(n) > 0$ for any n. Therefore, under the optimal DI system, all households with no labor supply of disabled agents receive positive amounts of DI benefits. However, the amount of DI benefits declines as able agent's ability n increases, or equivalently, as $z_0(n)$ increases. As noted, these properties make sense from the perspective of redistribution.

The third result of Proposition 3 states $dq^*/dn \geq 0$. Hence, the participation rate of disabled agents rises with the productivity of able agents. Intuitively, $\Delta U(n)$ increases with n under the optimal tax and DI system because $B(n)$ falls with n. Due to the increase in the utility gap, more disabled agents are induced to provide labor supply as n rises. Thus, the labor force participation rate of disabled agents tends to be higher for richer households.

The last result of the proposition is concerned with $\tau(n)$, which can be interpreted as the labor income tax rate of a disabled agent in a household with (n, q). To see why, recall that if a disabled agent supplies labor, the household income increases mechanically by w. However, the household income goes up only by $\Delta U(n) = U_1(n) - U_0(n)$ due to the income tax and corresponding adjustments in z_0 and z_1. In this sense, we can interpret $\Delta U(n)/w$ as the net-of-tax rate or the retention rate for the labor income of a disabled agent. Thus, the relationship between $\tau(n)$ and $\Delta U(n)$ can be formulated as

$$1 - \tau(n) = \frac{\Delta U(n)}{w} = \frac{C(q^*(n))}{w},$$

which is transformed to (22) in Proposition 3. From this equation, $d\tau/dn \leq 0$ follows because $dq^*/dn \geq 0$ and C is increasing.[10] This result is consistent other results in Proposition 3. As the optimal system provides more work incentives for disabled agents in rich households or high n households, the labor income tax rate for disabled agents should also decline with n.

[10]It is also worth our attention that $\tau(n) \leq 1$ because $C(q) \geq 0$ but $\tau(n) < 0$ is possible if $C(q^*(n)) > w$.

5 Validity of key assumptions

In the previous section, I derive the theoretical properties of the optimal income tax and DI system based on several assumptions. However, one may wonder how reasonable those assumptions are. If they are unlikely to hold for reasonablly calibrated $m(k)$, $v(k)$, $P(q)$, and $f(n)$ functions, the findings in Propositions 2 and 3 would be irrelevant for the tax and DI policy. Therefore, I examine the validity of Assumptions 1-4 here. For this purpose, I parameterize the model with functional forms that are widely used in economic analysis, and investigate whether or under what conditions Assumptions 1-4 are satisfied. Through this analysis, we can conclude that they are indeed quite reasonable.

Throughout this section, I use the following social welfare function.

$$\Psi(V) = V^{1-\theta}/(1-\theta), \ \theta > 0 \tag{23}$$

This function satisfies $\Psi' > 0$, $\Psi'' < 0$, and $\Psi''' > 0$, as assumed above. In this function, V is expressed in terms of goods and, therefore, Ψ should capture potential concavity in the household's utility function. Hence, $\theta \geq 1$ is likely in (23). Regarding $v(k)$, $m(k)$, and $P(q)$, we consider two cases: the CARA and CRRA parameterizations. In the CARA parameterization, all of the functions are assumed to exhibit constant absolute risk aversion, and in the CRRA parameterization, all of them are assumed to exhibit constant relative risk aversion.

5.1 CARA parameterization

In the CARA case, I use the following functions for simulation.

$$m(k) = 1 - \exp(-\sigma k), \ \sigma > 0$$

$$v(k) = a[\exp(\gamma k) - 1], \ \gamma > 0, \ a > 0$$

$$P(q) = 1 - \exp(-\eta q), \ \eta > 0$$

Note that all of them are in CARA forms. I also assume that both k and q are unbounded. It is easy to see that these functions satisfy all assumptions in Section 2: $m(0) = 0, \ m' >$

$0,\ m'' < 0,\ \max_{k \geq 0} m(k) \leq 1$ and $v(0) = 0,\ v' > 0,\ v'' > 0$. For these functions, household's FOC (8) is written as

$$q \leq \frac{v'(k_1)}{m'(k_1)} = \frac{a\gamma}{\sigma} \exp\left[(\gamma + \sigma) k_1\right] \text{ with equality for } k_1 > 0.$$

This condition implies that $k_1 = \frac{1}{\sigma+\gamma}(\ln q - \ln \chi)$ if $q > \chi = \frac{a\gamma}{\sigma}$ and $k_1 = 0$ otherwise. Hence, only agents with serious disability $q > \chi$ receive intra-household care in the CARA parameterization. By (11), we can also obtain the formula for $C(q)$ as follows.

$$C(q) = \begin{cases} q & \text{if } q \leq \chi \\ Aq^{\rho} - a & \text{if } q > \chi \end{cases}$$

where $\rho \equiv \frac{\gamma}{\gamma+\sigma} \in (0, 1)$ and $A \equiv \chi^{1-\rho} + a\chi^{-\rho}$. With this function, I examine the validity of Assumptions 2-4.

5.1.1 Examining key assumptions

Assumption 2 For Assumption 2, $\Psi'(U_1 - C(q))$ should be convex in q. Given the functional forms of $\Psi(V)$ and $C(q)$, we can show that the second derivative of $\Psi'(U_1 - C(q))$ is positive if and only if

$$G(q) \equiv (1 + \theta) - \left[\frac{U_1 - C(q)}{C'(q)\, q}\right] \left(-\frac{C''(q)\, q}{C'(q)}\right) > 0. \tag{24}$$

Note that $U_1 - C(q) \geq 0$ because Ψ allows only non-negative values. First, Assumption 2 is trivially satisfied for $q \leq \chi$ because $C''(q) = 0$ makes $G(q)$ always positive. Second, for two-earner households with $q > \chi$, $G(q)$ is rewritten as

$$G(q) = (1 + \theta) - \left(\frac{U_1 + a - Aq^{\rho}}{A\rho q^{\rho}}\right)(1 - \rho).$$

Since $G(q)$ increases with q and decreases with U_1, (24) holds true for any (n, q) if it holds for $U_1(\bar{n})$ and $q = \chi$.

$$(1 + \theta) > \left(\frac{U_1(\bar{n}) + a - A\chi^{\rho}}{A\rho\chi^{\rho}}\right)(1 - \rho)$$

According to simulations with various parameter values, this inequality tends to hold true for most parameter values if $\theta \geq 1$ and $U_1(\bar{n})$ is not too large.

Assumption 3 As $P(q) = 1 - \exp(-\eta q)$, $\eta > 0$, Assumption 3 requires

$$\frac{qp(q)}{P(q)} = \frac{q\eta \exp(-\eta q)}{1 - \exp(-\eta q)} = \frac{\eta q}{\exp(\eta q) - 1} \leq 1,$$

Because $\eta q \leq \exp(\eta q) - 1$ for any $q \geq 0$, Assumption 3 is always satisfied regardless of η.

Assumption 4 Using the formulae for $C(q)$ and $P(q)$, we can calculate $M(q)$ in Assumption 4 as the following.

$$M(q) = \begin{cases} \eta \frac{w-q}{1-\exp(-\eta q)} & \text{if } q \leq \chi \\ \frac{\eta}{A\rho} \frac{(w+a)q^{1-\rho} - Aq}{1-\exp(-\eta q)} & \text{if } q > \chi \end{cases}$$

If $q \leq \chi$, or for households with $k_1 = 0$, $M(q)$ declines with q because the numerator is decreasing whereas the denominator is increasing. Hence, Assumption 4 is satisfied in this case. If $q > \chi$, the denominator of $M(q)$ is an increasing exponential function whereas the numerator is a polynomial function which is increasing for small q but decreasing for large q. Thus, $M(q)$ tends to fall with q because the growth of its denominator tends to dominate that of its numerator. Therefore, Assumption 4 is likely to be satisfied for $q > \chi$. Combining these results, we can conclude that Assumption 4 tends to hold true.

5.2 CRRA parameterization

In the CRRA case, I use the following functions.

$$m(k) = \frac{b}{1-\sigma} k^{1-\sigma}, \ \sigma \in (0,1), \ b > 0,$$

$$v(k) = \frac{a}{1+\gamma} k^{1+\gamma}, \ \gamma > 0, \ a > 0.$$

$$P(q) = \left(\frac{q}{q_{\max}}\right)^{\eta}, \ \eta \in (0,1), \ q \leq q_{\max}$$

It is easy to see that these functions satisfy all the assumptions made in Section 2, except for $\max_{k \geq 0} m(k) \leq 1$. Since $m(k) = 1$ if $k = \bar{k} \equiv \left(\frac{1-\sigma}{b}\right)^{\frac{1}{1-\sigma}}$, I will set the range of q so that $k_1 \leq \bar{k}$ and $m(k_1) \leq 1$. For that purpose, I rewrite the household FOC (8) as

$$q \leq \frac{v'(k_1)}{m'(k_1)} = \frac{a}{b} k_1^{\gamma+\sigma} \text{ with equality for } k_1 > 0.$$

From this condition, we can obtain k_1 as follows.

$$k_1 = \left(\frac{b}{a}q\right)^{\frac{1}{\gamma+\sigma}}$$

Thus, $q \le \frac{a}{b}\left(\frac{1-\sigma}{b}\right)^{\frac{\gamma+\sigma}{1-\sigma}}$ is required to make $k_1 \le \bar{k}$. Hence, I assume that $q \le q_{\max} \le \frac{a}{b}\left(\frac{1-\sigma}{b}\right)^{\frac{\gamma+\sigma}{1-\sigma}}$. Using the functions of m and v, we can find

$$C\left(q\right) = q - Aq^{\rho},$$

where $\rho \equiv \frac{1+\gamma}{\sigma+\gamma} > 1$ and $A \equiv \frac{b^{\rho}}{a^{\rho-1}}\frac{\gamma+\sigma}{(1-\sigma)(1+\gamma)}$. With this $C\left(q\right)$ and $P\left(q\right)$ above, let us examine Assumptions 2-4 and the condition (21).

5.2.1 Examining key assumptions

Assumption 2 I show above that Assumption 2 is equivalent to (24). Using $C\left(q\right)$ above, $G\left(q\right)$ in (24) is calculated as

$$G\left(q\right) = \left(1+\theta\right) - A\rho\left(\rho-1\right)\frac{U_1 q^{\rho-2} - q^{\rho-1} + Aq^{2\rho-2}}{1 - 2A\rho q^{\rho-1} + A^2\rho^2 q^{2\rho-2}}.$$

To characterize this function, it is informative to calculate $\lim_{q\to 0} G\left(q\right)$ and $\lim_{q\to\infty} G\left(q\right)$. The condition $\rho > 1$ implies $2\rho - 2 > \rho - 1 > \rho - 2$. Thus, by comparing the powers of the terms in $G\left(q\right)$, we can conclude $\lim_{q\to\infty} G\left(q\right) = \theta + \frac{1}{\rho} > 0$ and

$$\lim_{q\to 0} G\left(q\right) = \left(1+\theta\right) - A\rho\left(\rho-1\right)U_1 \lim_{q\to 0} q^{\rho-2},$$

which implies $\lim_{q\to 0} G\left(q\right)$ is finite if $\rho \ge 2$ but $\lim_{q\to 0} G\left(q\right) = -\infty$ if $\rho < 2$. Hence, if $\rho < 2$, it is impossible for $G\left(q\right)$ to be always positive, whereas if $\rho \ge 2$, $G\left(q\right)$ tends to be U-shaped and $G\left(q\right) > 0$ is possible for sufficiently large θ. Therefore, Assumption 2 can be satisfied under two conditions: (i) $\rho \ge 2$, or equivalently, $2\sigma + \gamma \le 1$, and (ii) θ is sufficiently large.

Assumption 3 As $P\left(q\right) = \left(q/q_{\max}\right)^{\eta}$ with $\eta > 0$, Assumption 3 is equivalent to $\eta \le 1$ because

$$\frac{qp\left(q\right)}{P\left(q\right)} = \frac{q\eta q^{\eta-1}\left(q_{\max}\right)^{-\eta}}{q^{\eta}\left(q_{\max}\right)^{-\eta}} = \eta \le 1.$$

In fact, this is the assumption made by Kleven, Kreiner, and Saez (2009).

Assumption 4 With the CRRA functions, $M(q)$ in Assumption 4 is calculated as

$$M(q) = \eta \left(\frac{w - q + Aq^{\rho}}{q - A\rho q^{\rho}} \right) \left[\frac{(q_{\max})^{\eta}}{(q_{\max})^{\eta} - q^{\eta}} \right].$$

To understand its properties, let us examine the limiting values of $M(q)$. First, $\lim_{q\to 0} M(q) = \infty$ because the denominator in parentheses is zero. To find $\lim_{q\to q_{\max}} M(q)$, note the expression in square brackets explodes to infinity as q goes to $q_{\max}$. As for the expression in parentheses, the denominator $q_{\max} - A\rho (q_{\max})^{\rho}$ is finite and positive because it is $C'(q_{\max}) q_{\max}$ and $C'(q_{\max}) \in (0,1)$ by Lemma 2. Therefore, if the numerator $w - C(q_{\max}) = w - q_{\max} + A(q_{\max})^{\rho}$ is positive, then $\lim_{q\to q_{\max}} M(q) = \infty$. In this case, $M(q)$ cannot be decreasing for all q because $\lim_{q\to 0} M(q) = \lim_{q\to\infty} M(q) = \infty$. By contrast, if $w - C(q_{\max}) < 0$, $M(q)$ can be decreasing for all q since $\lim_{q\to 0} M(q) = \infty$ and $\lim_{q\to\infty} M(q) = -\infty$. Indeed, simulations indicate that $M(q)$ is diminishing for all q under the assumption $w < C(q_{\max})$. Hence, it can ensure Assumption 4 to be satisfied.

The assumption $w < C(q_{\max})$ is quite reasonable. In this model, w represents the average wage for disabled workers and $C(q_{\max})$ means the disability cost of labor for most severely disabled agents. As the average wage is unlikely to more than cover all the costs associated with disability for most severely disabled agents, $w < C(q_{\max})$ seems a realistic assumption. Hence, Assumption 4 also is likely to be satisfied in reality with the CRRA functions.

6 Concluding remarks

To be written

References

[1] Autor, D., A. R. Kostøl, M. Mogstad, and B. Setzler, 2017, "Disability Benefits, Consumption Insurance, and Household Labor Supply," *Norges Bank Research Working Paper* No. 16.

[2] Blundell, R., L. Pistaferri, and I. Saporta-Eksten, 2016, "Consumption Inequality and Family Labor Supply," *American Economic Review*, Vol. 106, No. 2, pp. 387-435.

[3] Charles, K. K., 1999, "Sickness in the Family: Health Shocks and Spousal Labor Supply," Working Paper, University of Michigan.

[4] Coile, C. C., 2004, "Health Shocks and Couples' Labor Supply Decisions," *NBER Working Paper* No. 10810.

[5] Cullen, J. B. and J. Gruber, 2000, "Does Unemployment Insurance Crowd out Spousal Labor Supply?" *Journal of Labor Economics*, Vol. 18, No. 3, pp. 546-572.

[6] Diamond, P. A. and J. A. Mirrlees, 1978, "A Model of Social Insurance with Variable Retirement," *Journal of Public Economics*, Vol. 10, No. 3, pp. 295-336.

[7] Fadlon, I. and T. H. Nielsen, 2015, "Household Responses to Severe Health Shocks and the Design of Social Insurance," *NBER Working Paper* No. 21352.

[8] Gallipoli, G. and L. Turner, 2011, "Household Responses to Individual Shocks: Disability and Labor Supply," Working Paper.

[9] Golosov, M. and A. Tsyvinski, 2006, "Designing Optimal Disability Insurance: A Case for Asset Testing," *Journal of Political Economy*, Vol. 114, No. 2, pp. 257-279.

[10] Haan, P. and V. Prowse, 2017, "Optimal Social Assistance and Unemployment Insurance in a Life-Cycle Model of Family Labor Supply and Savings," Working Paper.

[11] Kleven, H. J., C. T. Kreiner, and E. Saez, 2009, "The Optimal Income Taxation of Couples," *Econometrica*, Vol. 77, No. 2, pp. 537–560

[12] Lee, K., 2015, "Optimal Disability Insurance with Unobservable Skill Heterogeneity," *Journal of Public Economics*, Vol. 122, pp. 94-109.

[13] Low, H. and L. Pistaferri, 2015, "Disability Insurance and the Dynamics of the Incentive Insurance Trade-Off," *American Economic Review*, Vol. 105, No. 10, pp. 2986-3029.

[14] Meyer, B. D. and W. K. C. Mok, 2013, "Disability, Earnings, Income, and Consumption," *NBER Working Paper* No. 18869.

[15] Mirrlees, J. A., 1971, "An Exploration in the Theory of Optimum Income Taxation," *Review of Economic Studies*, Vol. 38, No. 2, pp. 175-208.

[16] Piketty, T. and E. Saez, 2013, "Optimal Labor Income Taxation," in A. J. Auerbach, R. Chetty, M. Feldstein, and E. Saez (eds.), *Handbook of Public Economics*, Vol. 5, Chapter 7, pp. 391-474.

[17] Saez, E., 2001, "An Exploration in the Theory of Optimum Income Taxation," *Review of Economic Studies*, Vol. 68, pp. 205-229.

[18] Stephens, M., 2002, "Worker Displacement and the Added Worker Effect," *Journal of Labor Economics*, Vol. 20, No. 3, pp. 504-537

A Appendix

A.1 Proof of Proposition 2

To begin with, the GBC (17) can be rewritten as

$$\int_{\underline{n}}^{\bar{n}} \left[\left\{ z_1 + w - h\left(\frac{z_1}{n}\right) - U_1 \right\} P\left(q^*\right) + \left\{ z_0 - h\left(\frac{z_0}{n}\right) - U_0 \right\} \left\{1 - P\left(q^*\right)\right\} \right] f\left(n\right) dn \geq 0 \quad (25)$$

using (10).[11] Let λ, $\zeta\left(n\right)$, and $\mu_l\left(n\right)$, respectively, be multipliers for (25), (13), and (18). Then, I can write the Hamiltonian for the government's problem as

$$\begin{aligned} H\left(n\right) \quad = \quad & \left[\int_0^{q^*} \Psi\left(U_1 - C\left(q\right)\right) p\left(q\right) dq + \int_{q^*}^{\infty} \Psi\left(U_0\right) p\left(q\right) dq \right] f\left(n\right) \\ & + \lambda \left[\left\{ z_1 + w - h\left(\frac{z_1}{n}\right) - U_1 \right\} P\left(q^*\right) + \left\{ z_0 - h\left(\frac{z_0}{n}\right) - U_0 \right\} \left\{1 - P\left(q^*\right)\right\} \right] f\left(n\right) \\ & + \zeta\left(n\right) \left[U_1 - U_0 - C\left(q^*\right)\right] \\ & + \mu_1\left(n\right) \frac{z_1}{n^2} h'\left(\frac{z_1}{n}\right) + \mu_0\left(n\right) \frac{z_0}{n^2} h'\left(\frac{z_0}{n}\right). \end{aligned}$$

The FOCs w.r.t. U_l, z_l, and q^* are given as follows.

$$\frac{\partial H}{\partial U_1} = \int_0^{q^*} \Psi'\left(U_1 - C\left(q\right)\right) p\left(q\right) dq f\left(n\right) - \lambda P\left(q^*\right) f\left(n\right) + \zeta\left(n\right) = -\dot{\mu}_1\left(n\right) \quad (26)$$

$$\frac{\partial H}{\partial U_0} = \int_{q^*}^{\infty} \Psi'\left(U_0\right) p\left(q\right) dq f\left(n\right) - \lambda \left[1 - P\left(q^*\right)\right] f\left(n\right) - \zeta\left(n\right) = -\dot{\mu}_0\left(n\right) \quad (27)$$

$$\frac{\partial H}{\partial z_1} = \lambda \left[1 - \frac{1}{n} h'\left(\frac{z_1}{n}\right) \right] P\left(q^*\right) f\left(n\right) + \mu_1\left(n\right) \left\{ \frac{1}{n^2} h'\left(\frac{z_1}{n}\right) + \frac{z_1}{n^3} h''\left(\frac{z_1}{n}\right) \right\} = 0 \quad (28)$$

$$\frac{\partial H}{\partial z_0} = \lambda \left[1 - \frac{1}{n} h'\left(\frac{z_0}{n}\right) \right] \left[1 - P\left(q^*\right)\right] f\left(n\right) + \mu_0\left(n\right) \left\{ \frac{1}{n^2} h'\left(\frac{z_0}{n}\right) + \frac{z_0}{n^3} h''\left(\frac{z_0}{n}\right) \right\} = 0 \quad (29)$$

$$\frac{\partial H}{\partial q^*} = \lambda \left[T_1\left(z_1\right) - T_0\left(z_0\right)\right] p\left(q^*\right) f\left(n\right) - \zeta\left(n\right) C'\left(q^*\right) = 0 \quad (30)$$

and the following transversality conditions:

$$\mu_l\left(\bar{n}\right) U_l\left(\bar{n}\right) = \mu_l\left(\underline{n}\right) U_l\left(\underline{n}\right) = 0 \text{ for } l \in \{0, 1\} \quad (31)$$

Along with the FOCs, we also need conditions under which control variables $\left(z_1, z_0, q^*\right)$ are maximizers almost everywhere in n. Hence, the Hessian of the Hamiltonian with respect

[11] Throughout the appendix, I suppress variables' dependence on n or n' unless it causes confusion.

to (z_1, z_0, q^*) should be negative definite. To see this, let H_{xy} denote the second derivative of H with respect to x and y. Using $H_{z_1 z_0} = 0$, we can write the Hessian as follows.

$$D^2 H = \begin{bmatrix} H_{z_1 z_1} & 0 & H_{z_1 q^*} \\ 0 & H_{z_0 z_0} & H_{z_0 q^*} \\ H_{z_1 q^*} & H_{z_0 q^*} & H_{q^* q^*} \end{bmatrix}$$

The usual conditions for the Hessian to be negative definite imply $H_{z_1 z_1} < 0$, $H_{z_0 z_0} < 0$, and $H_{q^* q^*} < 0$. By the definition of $\Lambda(z)$ in Assumption 1, we can rewrite (28) as

$$H_{z_1} = \mu_1 \left[\frac{1}{n^2} h'\left(\frac{z_1}{n}\right) + \frac{z_1}{n^3} h''\left(\frac{z_1}{n}\right) \right] \left[1 + \frac{\lambda P(q^*) f(n) n}{\mu_1} \Lambda(z_1) \right] = 0,$$

which implies $1 + \frac{\lambda P(q^*) f(n) n}{\mu_1} \Lambda(z_1) = 0$ because the terms in the first square brackets are strictly positive. Using this result, we can show $H_{z_1 z_1} < 0$ because

$$\begin{aligned} H_{z_1 z_1} &= \mu_1 \left[1 + \frac{\lambda P(q^*) f(n) n}{\mu_1} \Lambda(z_1) \right] \frac{\partial}{\partial z_1} \left[\frac{1}{n^2} h'\left(\frac{z_1}{n}\right) + \frac{z_1}{n^3} h''\left(\frac{z_1}{n}\right) \right] + \\ & \quad \mu_1 \left[\frac{1}{n^2} h'\left(\frac{z_1}{n}\right) + \frac{z_1}{n^3} h''\left(\frac{z_1}{n}\right) \right] \frac{\partial}{\partial z_1} \left[1 + \frac{\lambda P(q^*) f(n) n}{\mu_1} \Lambda(z_1) \right] \\ &= \left[\frac{1}{n^2} h'\left(\frac{z_1}{n}\right) + \frac{z_1}{n^3} h''\left(\frac{z_1}{n}\right) \right] \lambda P(q^*) f(n) n \Lambda'(z_1) < 0, \end{aligned}$$

where $\Lambda'(z_1) < 0$ by Assumption 1. By the same logic, Assumption 1 also ensures

$$H_{z_0 z_0} = \left[\frac{1}{n^2} h'\left(\frac{z_0}{n}\right) + \frac{z_0}{n^3} h''\left(\frac{z_0}{n}\right) \right] \lambda \left[1 - P(q^*)\right] f(n) n \Lambda'(z_0) < 0.$$

Finally, $H_{q^* q^*} < 0$ requires

$$H_{q^* q^*} = \lambda \left[T_1(z_1) - T_0(z_0)\right] p'(q^*) f(n) - \zeta(n) C''(q^*) < 0.$$

From this condition, we obtain (21) using $\lambda \left[T_1(z_1) - T_0(z_0)\right] p(q^*) f(n) = \zeta(n) C'(q^*)$ by (30).

To drive the optimal tax formulae, I use (4) and (6) to rewrite (28) and (29) as

$$\mu_1(n) = -\lambda n \frac{\left[1 - \frac{1}{n} h'\left(\frac{z_1}{n}\right)\right] P(q^*) f(n)}{\frac{1}{n} h'\left(\frac{z_1}{n}\right) \left[1 + \frac{z_1}{n} h''\left(\frac{z_1}{n}\right) / h'\left(\frac{z_1}{n}\right)\right]} = -\lambda P(q^*) f(n) n \frac{T_1'}{1 - T_1'} \frac{1}{1 + 1/\varepsilon_1}, \tag{32}$$

$$\mu_0(n) = -\lambda n \frac{\left[1 - \frac{1}{n} h'\left(\frac{z_0}{n}\right)\right] \left[1 - P(q^*)\right] f(n)}{\frac{1}{n} h'\left(\frac{z_0}{n}\right) \left[1 + \frac{z_0}{n} h''\left(\frac{z_0}{n}\right) / h'\left(\frac{z_0}{n}\right)\right]} = -\lambda \left[1 - P(q^*)\right] f(n) n \frac{T_0'}{1 - T_0'} \frac{1}{1 + 1/\varepsilon_0}. \tag{33}$$

If we impose relevant individual rationality conditions, $U_l(\bar{n})$ and $U_l(\underline{n})$ are positive. Then, by (31),

$$\mu_l(\bar{n}) = \mu_l(\underline{n}) = 0, \text{ for } l = 0, 1,$$

which, by (32) and (33), means

$$T_l'(z_l(\bar{n})) = T_l'(z_l(\underline{n})) = 0 \text{ for } l = 0, 1. \tag{34}$$

Also, we can solve (30) for $\zeta(n)$ as

$$\zeta(n) = \lambda \left[\frac{T_1(z_1) - T_0(z_0)}{C'(q^*)}\right] p(q^*) f(n). \tag{35}$$

Integrating (26) and (27) from n to $\bar{n}$ yields

$$-\left[\mu_1(n')\right]_n^{\bar{n}} = \mu_1(n) = \int_n^{\bar{n}} \left[\int_0^{q^*} \Psi'(U_1 - C(q)) p(q) dq f(n') - \lambda P(q^*) f(n') + \zeta(n')\right] dn',$$

$$-\left[\mu_0(n')\right]_n^{\bar{n}} = \mu_0(n) = \int_n^{\bar{n}} \left[\int_{q^*}^{\infty} \Psi'(U_0) p(q) dq f(n') - \lambda[1 - P(q^*)] - \zeta(n')\right] dn'.$$

Substituting (35) into these equations yields

$$\mu_1(n) = \int_n^{\bar{n}} \left[\frac{\int_0^{q^*} \Psi'(U_1 - C(q)) p(q) dq}{\lambda P(q^*)} \lambda P(q^*) - \lambda P(q^*) + \lambda \left[\frac{T_1(z_1) - T_0(z_0)}{C'(q^*)}\right] p(q^*)\right] f(n') dn',$$

$$\mu_0(n) = \int_n^{\bar{n}} \left[\frac{\int_{q^*}^{\infty} \Psi'(U_0) p(q) dq}{\lambda[1 - P(q^*)]} \lambda[1 - P(q^*)] - \lambda[1 - P(q^*)] - \lambda \left[\frac{T_1(z_1) - T_0(z_0)}{C'(q^*)}\right] p(q^*)\right] f(n') dn'.$$

Using the definition of g_l, these equations can be re-expressed as

$$\mu_1(n) = -\lambda \int_n^{\bar{n}} \left[(1 - g_1) P(q^*) - \frac{T_1(z_1) - T_0(z_0)}{C'(q^*)} p(q^*)\right] f(n') dn',$$

$$\mu_0(n) = -\lambda \int_n^{\bar{n}} \left[(1 - g_0)[1 - P(q^*)] + \frac{T_1(z_1) - T_0(z_0)}{C'(q^*)} p(q^*)\right] f(n') dn'.$$

Equating these equations to (32) and (33), I obtain the optimal tax formulae:

$$\frac{T_1'}{1 - T_1'} = \left(1 + \frac{1}{\varepsilon_1}\right) \frac{1}{P(q^*) f(n) n} \int_n^{\bar{n}} \left[(1 - g_1) P(q^*) - \frac{T_1(z_1) - T_0(z_0)}{C'(q^*)} p(q^*)\right] f(n') dn',$$

$$\frac{T_0'}{1 - T_0'} = \left(1 + \frac{1}{\varepsilon_0}\right) \frac{1}{[1 - P(q^*)] f(n) n} \int_n^{\bar{n}} \left[(1 - g_0)[1 - P(q^*)] + \frac{T_1(z_1) - T_0(z_0)}{C'(q^*)} p(q^*)\right] f(n') dn'.$$

A.2 Proof of Proposition 3

Result 1: $B'(n) \leq 0$ or $T_1' \leq T_0'$ for all n

I prove this by contradiction. Suppose $T_1' > T_0'$ for some n. Notice first that $T_0' = T_1' = 0$ at both $\bar{n}$ and $\underline{n}$ because $T_l'(z_l(n))$ is proportional to $\mu_l(n)$ by (28) and (29), and $\mu_l(\bar{n}) = \mu_l(\underline{n}) = 0$ by the transversality conditions (31). Since both T_1' and T_0' are continuous, and $T_0' = T_1' = 0$ at both $\bar{n}$ and $\underline{n}$, there should be an interval (n_a, n_b) for which $T_1' > T_0'$ and $T_1' = T_0'$ for n_a and n_b. Then $z_1 < z_0$ on (n_a, n_b) and $z_1 = z_0$ for n_a and n_b due to the FOCs of the household problem $1 - T_l' = \frac{1}{n} h'\left(\frac{z_l}{n}\right)$. Since $z_1 < z_0$ over (n_a, n_b), $U_1' < U_0'$ because of the envelope condition $U_l'(n) = \frac{z_l}{n^2} h'\left(\frac{z_l}{n}\right)$. Using this result, we obtain $q^*(n_a) > q^*(n_b)$ because

$$\frac{dq^*}{dn} = \frac{U_1' - U_0'}{C'(q^*)} < 0 \text{ over } (n_a, n_b).$$

Moreover, as $z_1 = z_0$ at n_a and n_b,

$$U_1(n_a) - U_0(n_a) = w - \Delta T(n_a) = C(q^*(n_a)),$$

$$U_1(n_b) - U_0(n_b) = w - \Delta T(n_b) = C(q^*(n_b)),$$

where $\Delta T(n) = T_1(z_1(n)) - T_0(z_0(n))$. Because $q^*(n_a) > q^*(n_b)$ and $C' > 0$, we obtain $\Delta T(n_a) < \Delta T(n_b)$.

On the other hand, notice that

$$\frac{T_l'}{1 - T_l'} \frac{\varepsilon_l}{1 + \varepsilon_l} = \frac{1 - \frac{1}{n} h'}{\frac{1}{n} h'} \frac{h'}{\frac{z_l}{n} h'' + h'} = \frac{1 - \frac{1}{n} h'}{\frac{z_l}{n^2} h'' + \frac{1}{n} h'}.$$

Hence, by Assumption 1 and $z_1 < z_0$ on (n_a, n_b) and $z_1 = z_0$ for n_a and n_b,

$$\frac{T_1'}{1 - T_1'} \frac{\varepsilon_1}{1 + \varepsilon_1} > \frac{T_0'}{1 - T_0'} \frac{\varepsilon_0}{1 + \varepsilon_0} \text{ on } (n_a, n_b),$$

$$\frac{T_1'}{1 - T_1'} \frac{\varepsilon_1}{1 + \varepsilon_1} = \frac{T_0'}{1 - T_0'} \frac{\varepsilon_0}{1 + \varepsilon_0} \text{ for } n_a \text{ and } n_b.$$

These equations can be rewritten using (19) and (20) as

$$\begin{aligned} \Omega_1(n) &> \Omega_0(n) \text{ for } n \in (n_a, n_b), \\ \Omega_1(n) &= \Omega_0(n) \text{ for } n = n_a, n_b, \end{aligned}$$

where

$$\Omega_1(n) \equiv \frac{1}{P(q^*)} \int_n^{\bar{n}} \left[(1-g_1) P(q^*) - \frac{\Delta T(n)}{C'(q^*)} p(q^*) \right] f(n')\, dn',$$

$$\Omega_0(n) \equiv \frac{1}{1-P(q^*)} \int_n^{\bar{n}} \left[(1-g_0) [1-P(q^*)] + \frac{\Delta T(n)}{C'(q^*)} p(q^*) \right] f(n')\, dn'.$$

These equations imply that the graphs of $\Omega_1(n)$ and $\Omega_0(n)$ cross each other at n_a and n_b, and that $\Omega_1'(n_a) > \Omega_0'(n_a)$ and $\Omega_1'(n_b) < \Omega_0'(n_b)$. At the end points, $T_1' = T_0'$, $z_1 = z_0$, and $U_1' = U_0'$. These conditions imply that $dq^*/dn = \frac{U_1'-U_0'}{C'(q^*)} = 0$, which in turn leads to $dP(q^*)/dn = 0$. Due to these results, $\Omega_1'(n_a) > \Omega_0'(n_a)$ and $\Omega_1'(n_b) < \Omega_0'(n_b)$ are equivalently written as

$$1 - g_1(n_a) - \frac{\Delta T(n_a)}{C'(q^*(n_a))} \frac{p(q^*(n_a))}{P(q^*(n_a))} < 1 - g_0(n_a) + \frac{\Delta T(n_a)}{C'(q^*(n_a))} \frac{p(q^*(n_a))}{1-P(q^*(n_a))},$$

$$1 - g_1(n_b) - \frac{\Delta T(n_b)}{C'(q^*(n_b))} \frac{p(q^*(n_b))}{P(q^*(n_b))} > 1 - g_0(n_b) + \frac{\Delta T(n_b)}{C'(q^*(n_b))} \frac{p(q^*(n_b))}{1-P(q^*(n_b))}.$$

These inequalities imply

$$g_0(n_a) - g_1(n_a) < \frac{\Delta T(n_a)}{C'(q^*(n_a))} \frac{p(q^*(n_a))}{[1-P(q^*(n_a))] P(q^*(n_a))}$$

$$g_0(n_b) - g_1(n_b) > \frac{\Delta T(n_b)}{C'(q^*(n_b))} \frac{p(q^*(n_b))}{[1-P(q^*(n_b))] P(q^*(n_b))}$$

From these inequalities, we obtain

$$\frac{\Delta T(n_a)}{C'(q^*(n_a))} \frac{p(q^*(n_a))}{[1-P(q^*(n_a))] P(q^*(n_a))} > g_0(n_a) - g_1(n_a) > g_0(n_b) - g_1(n_b) > \frac{\Delta T(n_b)}{C'(q^*(n_b))} \frac{p(q^*(n_b))}{[1-P(q^*(n_b))] P(q^*$$

where $g_0(n_a) - g_1(n_a) > g_0(n_b) - g_1(n_b)$ is formally proved in Lemma 3 below. Since $\Delta T = w - C(q^*)$ at n_a and n_b, the above inequality can be rewritten as

$$\frac{w - C(q^*(n_a))}{C'(q^*(n_a))} \frac{p(q^*(n_a))}{[1-P(q^*(n_a))] P(q^*(n_a))} > \frac{w - C(q^*(n_b))}{C'(q^*(n_b))} \frac{p(q^*(n_b))}{[1-P(q^*(n_b))] P(q^*(n_b))}.$$

By Assumption 4, this inequality implies $q^*(n_a) < q^*(n_b)$ and

$$\Delta T(n_a) = w - C(q^*(n_a)) > w - C(q^*(n_b)) = \Delta T(n_b).$$

This inequality, however, contradicts the result $\Delta T(n_a) < \Delta T(n_b)$ that I have shown above. This contradiction invalidates the initial supposition that $T_1' > T_0'$ for some n. Consequently, we can conclude that $T_1' \leq T_0'$ for all $n \in [\underline{n}, \bar{n}]$.

Result 2: $B(n) \geq B(\bar{n}) > 0$

Since $T_0' \geq T_1'$ on $(\underline{n}, \bar{n})$ and $T_0' = T_1'$ for $\underline{n}$ and $\bar{n}$, we have $z_1 \geq z_0$ on $(\underline{n}, \bar{n})$ and $z_1 = z_0$ for $\underline{n}$ and $\bar{n}$. By Assumption 1,

$$\frac{T_1'}{1-T_1'}\frac{\varepsilon_1}{1+\varepsilon_1} = \frac{1-\frac{1}{n}h_1'}{\frac{z_1}{n^2}h_1'' + \frac{1}{n}h_1'} \leq \frac{1-\frac{1}{n}h_0'}{\frac{z_0}{n^2}h_0'' + \frac{1}{n}h_0'} = \frac{T_0'}{1-T_0'}\frac{\varepsilon_0}{1+\varepsilon_0}$$

with equality for $\underline{n}$ and $\bar{n}$. Due to (19) and (20), this equation implies that $\Omega_0(n) \geq \Omega_1(n)$ on $(\underline{n}, \bar{n})$ and $\Omega_0(n) = \Omega_1(n)$ for $\underline{n}$ and $\bar{n}$. To satisfy these conditions, $\Omega_0'(\bar{n}) \leq \Omega_1'(\bar{n})$ should hold true by a similar argument as above. Using the conditions satisfied at $\bar{n}$, we have

$$1 - g_1(\bar{n}) - \frac{\Delta T(\bar{n})}{C'(q^*(\bar{n}))}\frac{p(q^*(\bar{n}))}{P(q^*(\bar{n}))} \leq 1 - g_0(\bar{n}) + \frac{\Delta T(\bar{n})}{C'(q^*(\bar{n}))}\frac{p(q^*(\bar{n}))}{1-P(q^*(\bar{n}))},$$

or equivalently,

$$g_0(\bar{n}) - g_1(\bar{n}) \leq \frac{\Delta T(\bar{n})}{C'(q^*(\bar{n}))}\frac{p(q^*(\bar{n}))}{[1-P(q^*(\bar{n}))]P(q^*(\bar{n}))}.$$

Therefore, $\Delta T(\bar{n}) > 0$ because $g_0(\bar{n}) - g_1(\bar{n}) > 0$.

To show $\Delta T(n) \geq \Delta T(\bar{n})$ for $n \leq \bar{n}$, notice that $T_0' \geq T_1'$, $z_0 \leq z_1$, and $U_0' \leq U_1'$ imply that $dq^*/dn \geq 0$. Also, for $n < \bar{n}$, $z_0 \leq z_1$ implies

$$\left(z_1 - h\left(\frac{z_1}{n}\right)\right) - \left(z_0 - h\left(\frac{z_0}{n}\right)\right) \geq 0$$

because $z - h\left(\frac{z}{n}\right)$ is an increasing function up to z such that $1 - \frac{1}{n}h' = 0$. Since z_1 and z_0 are determined by $1 - \frac{1}{n}h'\left(\frac{z_l}{n}\right) = T_l' \geq 0$, $z_1 \geq z_0$ leads to the above inequality. Combining the results so far, we have

$$\begin{aligned} & w - \Delta T(\bar{n}) = C(q^*(\bar{n})) \geq C(q^*(n)) = U_1(n) - U_0(n) \\ = \;& \left[\left(z_1 - h\left(\frac{z_1}{n}\right)\right) - \left(z_0 - h\left(\frac{z_0}{n}\right)\right)\right] + w - \Delta T(n) \geq w - \Delta T(n). \end{aligned}$$

Consequently, we conclude that $\Delta T(n) \geq \Delta T(\bar{n})$.

Result 3: $dq^*/dn \geq 0$

By the total differential of (13) and the envelope condition (18),

$$\frac{dq^*}{dn} = \frac{U_1'(n) - U_0'(n)}{C'(q^*(n))} = \frac{\frac{z_1}{n^2}h'\left(\frac{z_1}{n}\right) - \frac{z_0}{n^2}h'\left(\frac{z_0}{n}\right)}{C'(q^*(n))}.$$

We have $z_1 \geq z_0$ because $T_1' \leq T_0'$. Therefore, dq^*/dn is non-negative.

Result 4: $d\tau/dn \leq 0$

From the definition of $\tau(n)$,

$$\frac{d\tau}{dn} = \frac{U_0' - U_1'}{w}.$$

Since $z_1 \geq z_0$ and $U_1' \geq U_0'$ because $T_1' \leq T_0'$, we obtain $d\tau/dn \leq 0$.

A.3 Proof of Lemma 3

Lemma 3 *Under Assumptions 2 and 3, if $T_1' > T_0'$ on (n_a, n_b) with equality at end points,*

$$\frac{d\left[g_0(n) - g_1(n)\right]}{dn} < 0 \text{ on } (n_a, n_b),$$

and $g_0(n_a) - g_1(n_a) > g_0(n_b) - g_1(n_b)$.

Proof. By definition,

$$g_0(n) - g_1(n) = \frac{1}{\lambda}\left[\Psi'(U_0) - \frac{\int_0^{q^*} \Psi'(U_1 - C(q))\,p(q)\,dq}{P(q^*)}\right] > 0$$

Differentiating the equation w.r.t. n, we obtain

$$g_0'(n) = \frac{\Psi''(U_0)\,U_0'(n)}{\lambda},$$

$$g_1'(n) = \frac{\left\{\int_0^{q^*} \Psi''(U_1 - C(q))\,p(q)\,dq U_1'(n) + \Psi'(U_1 - C(q^*))\,p(q^*)\,\frac{dq^*}{dn}\right\} P(q^*)}{\lambda\left[P(q^*)\right]^2} - \frac{\int_0^{q^*} \Psi'(U_1 - C(q))\,p(q)\,dq\left\{p(q^*)\,\frac{dq^*}{dn}\right\}}{\lambda\left[P(q^*)\right]^2}$$

Rearranging terms, we obtain

$$\frac{d\left[g_0(n) - g_1(n)\right]}{dn} = \frac{\Psi''(U_0)\,U_0'(n)}{\lambda} - \frac{\int_0^{q^*} \Psi''(U_1 - C(q))\,p(q)\,dq U_1'(n)}{\lambda P(q^*)} + \frac{p(q^*)}{P(q^*)}\frac{dq^*}{dn}\left[g_1(n) - g_0(n)\right].$$

Because $dq^*/dn = \frac{U_1' - U_0'}{C(q^*)}$,

$$U_0' = U_1' - \frac{dq^*}{dn}C'(q^*).$$

Using this equation, we can rewrite $\frac{d[g_0(n) - g_1(n)]}{dn}$ as

$$\frac{d\left[g_0(n) - g_1(n)\right]}{dn} = A(n) + B(n),$$

where

$$A(n) = U_1' \left[\frac{\Psi''(U_0)}{\lambda} - \frac{\int_0^{q^*} \Psi''(U_1 - C(q)) p(q)\, dq}{\lambda P(q^*)} \right],$$

$$B(n) = \frac{dq^*}{dn} \left[-\{g_0(n) - g_1(n)\} \frac{p(q^*)}{P(q^*)} - \frac{\Psi''(U_0) C'(q^*)}{\lambda} \right]. \tag{36}$$

Regarding the sign of $A(n)$, notice that $U_1 - C(q) \geq U_0$ for $q \leq q^*$. Because Ψ is concave and Ψ' is convex,

$$\frac{\Psi''(U_0)}{\lambda} < \frac{\int_0^{q^*} \Psi''(U_1 - C(q)) p(q)\, dq}{\lambda P(q^*)} < 0.$$

By this inequality and $U_1' = \frac{z_1}{n^2} h'\left(\frac{z_1}{n}\right) > 0$, we conclude $A(n) < 0$. As for $B(n)$, notice first that $\frac{dq^*}{dn} = \frac{U_1' - U_0'}{C(q^*)} < 0$ because $z_0 > z_1$ and $U_0'(n) > U_1'(n) > 0$ due to the assumption $T_1' > T_0'$. In addition,

$$g_0(n) - g_1(n) < -\frac{\Psi''(U_0) C'(q^*) q^*}{\lambda}$$

because

$$\begin{aligned}
g_0(n) - g_1(n) &= \frac{\int_0^{q^*} [\Psi'(U_1 - C(q^*)) - \Psi'(U_1 - C(q))]\, dP(q)}{\lambda P(q^*)} \\
&= -\frac{\int_0^{q^*} [\Psi''(U_1 - C(\hat{q})) C'(\hat{q}) (q^* - q)]\, dP(q)}{\lambda P(q^*)} \text{ (by the mean value theorem)} \\
&< -\frac{\int_0^{q^*} [\Psi''(U_1 - C(\hat{q})) C'(\hat{q}) q^*]\, dP(q)}{\lambda P(q^*)} \; (\because \Psi''(U_1 - C(\hat{q})) C'(\hat{q}) q < 0) \\
&< -\frac{\Psi''(U_1 - C(q^*)) C'(q^*) q^*}{\lambda} \text{ (by Assumption 2)} \\
&= -\frac{\Psi''(U_0) C'(q^*) q^*}{\lambda}.
\end{aligned}$$

The second equality holds because by the mean value theorem, for any $q \in [0, q^*]$, there exists $\hat{q}$ such that

$$\Psi'(U_1 - C(q^*)) - \Psi'(U_1 - C(q)) = -\Psi''(U_1 - C(\hat{q})) C'(\hat{q}) (q^* - q).$$

As $\frac{q^* p(q^*)}{P(q^*)} \leq 1$ by Assumption 3, we multiply the inequality by $\frac{q^* p(q^*)}{P(q^*)}$ to obtain

$$[g_0(n) - g_1(n)] \frac{p(q^*)}{P(q^*)} < -\frac{\Psi''(U_0) C'(q^*)}{\lambda}.$$

Therefore, $B(n) < 0$ because $dq^*/dn < 0$ and the terms in the square brackets in (36) are positive. Since both $A(n)$ and $B(n)$ are negative, we can conclude $\frac{d[g_0(n) - g_1(n)]}{dn} < 0$, which implies $g_0(n_a) - g_1(n_a) > g_0(n_b) - g_1(n_b)$. ■

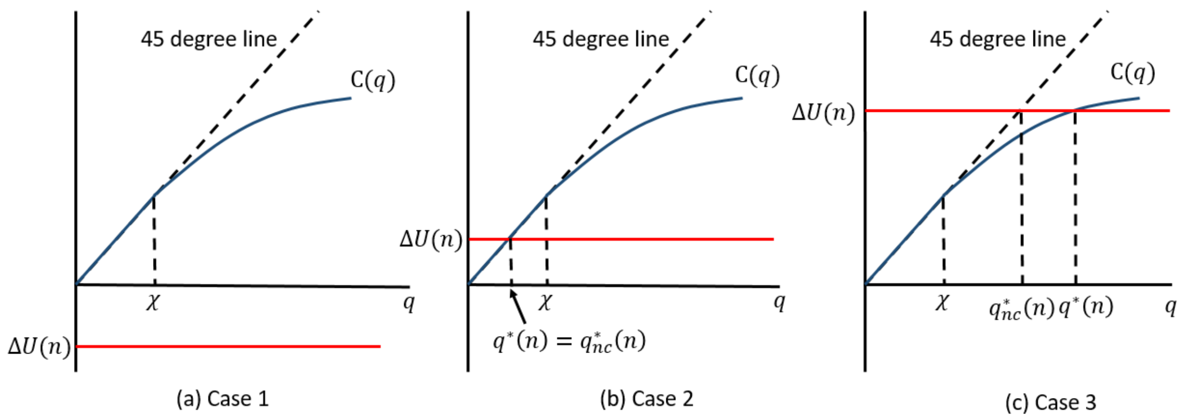

Figure 1: Labor supply of disabled agents. Each panel corresponds to a case in Proposition 1. In all panels, $\Delta U(n) = U_1(n) - U_0(n)$ and labor supply requires $\Delta U(n) \geq C(q)$. In panel (a), no q satisfies the condition as $\Delta U(n) \leq 0$. Hence, no disabled agents provide labor supply. In panel (b), $\Delta U(n) \leq \chi$ and $C(q) = q$ for q that satisfies $\Delta U(n) \geq C(q)$. Thus, $q^*(n) = q^*_{nc}(n)$, and intra-household care does not affect the labor force participation rate of disabled agents. In panel (c), $C(q) < q$ for some q that satisfies $\Delta U(n) \geq C(q)$ because $\Delta U(n) > \chi$. Thus, $q^*(n) > q^*_{nc}(n)$, and intra-household care raises the labor force participation rate of disabled agents.