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November 9, 2019

Abstract

This paper explores the optimal design of biased contests. A designer imposes an identity-dependent treatment on contestants, which varies the balance of the playing field. A generalized lottery contest typically yields no closed-form equilibrium solutions, which nullifies the usual implicit programming approach to optimal contest design and limits analysis to restricted settings. We propose an alternative approach that allows us to circumvent this difficulty and characterize the optimum in a general setting under a wide array of objective functions without solving for the equilibrium explicitly. Our technique applies to broad contexts, and the analysis it enables generates novel insights into incentive provision in contests and their optimal design. For instance, we demonstrate that the conventional wisdom of leveling the playing field, which is obtained in limited settings in previous studies, does not generally hold.

Keywords: Contest Design; Optimal Biases; Tullock Contest.

JEL Classification Codes: C72; D72.

*We are grateful to Coeditor Dilip Mookherjee and four anonymous reviewers for constructive and insightful comments on earlier versions of this paper. We thank Kyung Hwan Baik, Stefano Barbieri, Ulrich Doraszelski, Mikhail Drugov, Jörg Franke, Alex Gershkov, Simon Grant, Eun Jeong Heo, Ju Hu, Kai Konrad, Takashi Kunimoto, Wolfgang Leininger, Jianpei Li, Bin Liu, Qingmin Liu, Tore Nilssen, Paul Schweinzer, Xi Weng, Jingyi Xue, Lixin Ye, Jun Yu, and Junjie Zhou, and seminar/conference participants at the Chinese University of Hong Kong (Shenzhen), Singapore Management University, Sun Yat-sen University, Sungkyunkwan University, University of International Business and Economics, the 4th Annual Conference on Contests: Theory and Evidence, and the 2018 International Conference on Economic Theory and Applications (Chengdu) for helpful discussions, suggestions, and comments. We thank Shanglyu Deng and Yuxuan Zhu for excellent research assistance. Wu thanks the National Natural Science Foundation of China (No. 71803003) and the seed fund of the School of Economics, Peking University, for financial support. Any errors are our own.

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1 Introduction

Contests are widely administered in practice to mobilize productive efforts. For instance, the internal labor market inside a firm largely resembles a contest, in which workers strive to be promoted to higher rungs on hierarchical ladders (see, for instance, Rosen, 1986). Governments, firms, and even wealthy individuals sponsor innovation contests to secure technological solutions or promote focused research efforts for valuable projects (see Che and Gale, 2003). In a contest, contenders expend costly effort to vie for limited prizes and are rewarded based on their relative performance instead of absolute output metrics. The economics literature has long recognized this simple mechanism as a convenient remedy for the pervasive moral hazard problem. Lazear and Rosen (1981) and Rosen (1986), among others, propose the celebrated thesis that contests could achieve the same level of efficiency as an incentive contract.

The ubiquity of contest-like competitive activities has triggered broad interest in their strategic substance and the optimal design of competitive schemes that spur incentive provision.[1] In this paper, we explore a classic question in the contest literature: How should a designer bias the competition in a contest to boost its performance? Contestants' strategic behaviors sensitively depend on their relative competitiveness. This can exogenously be determined by contestants' innate strength—e.g., prize valuations—and various environmental factors of the competition, e.g., home court advantage in sports and litigation. It can also be determined endogenously by the choice of contest rules. A designer can impose identity-dependent preferential treatments on contestants—tailored to their individual characteristics—to vary contestants' relative standing. Consider, for instance, government policies that favor small and medium-sized enterprises (SMEs) in public procurement to support local entrepreneurship (Che and Gale, 2003; Epstein, Mealem and Nitzan, 2011) and colleges that allocate bonus points to minority applicants (Fu, 2006; Franke, 2012). In CEO succession races, the leading candidate is often appointed president or chief operating officer (COO) of the firm: The key appointment endows him/her with superior corporate resources, which boosts the candidate's productivity relative to others (Fu and Wu, 2019b).

The literature broadly embraces the notion that a more level playing field fuels more competitions and incentivizes contestants.[2,3] The conventional wisdom, however, must be examined more comprehensively, as the vast literature on optimal biased contests has typi-

[1]See Fu and Wu (2019a) for a recent survey of theoretical studies of contests.

[2]See the recent survey of Chowdhury, Esteve-González and Mukherjee (2019) on biased contests.

[3]Two notable exceptions are provided by Fu, Lu and Lu (2012) and Drugov and Ryvkin (2017). The former show that a performance-maximizing administrator may allocate more productive resources to an ex ante stronger firm. The latter show that it can be optimal to bias an otherwise symmetric contest. Both studies focus on two-player settings.

cally focused on relatively restricted settings—e.g., two players, stylized contest technologies, and limited objective functions—due to technical challenges. This paper develops a novel optimization approach that allows us to circumvent the analytical difficulty and identify the key properties of the optimum in broad contexts. Our analysis illuminates the nature of incentive provision in contests and yields novel implications that could refute the conventional wisdom.

Nature of the Generalized Optimization Problem

The conventional wisdom of leveling the playing field is underpinned primarily by the rationale that favoring the underdog boosts his incentive, which further deters the favorite from slackening off. This logic, however, rests on contestants' nonmonotone best responses in bilateral strategic relation (Lazear and Rosen, 1981; Dixit, 1987). Involving more than two players fundamentally alters the nature of the strategic interaction in a contest and its optimal design.

First, setting optimal identity-dependent preferential treatments in a two-player setting is a unidimensional problem, because favoring one equivalently handicaps the other. With more than two contestants, the strategic interactions are no longer reciprocal or direct; instead, contestants are entangled in an intricately reflexive network, which expands the channels through which a treatment could manipulate their behavior.

Imagine a contest with three players who are indexed by 1, 2, and 3. Suppose that favoritism is awarded to player 3. This directly affects his own incentive, which further compels the other two to vary their effort choices in response. The favoritism given to player 3, however, also affects the strategic interaction between players 1 and 2: Player 1's response to the change in player 3's effort forces player 2 to adjust his behavior, and vice versa. The incentive effect of the favoritism awarded to player 3 is compounded by the interaction between players 1 and 2, which does not exist in a bilateral competition and obscures the role played by the treatment: Its overall effect must sum up contestants' strategic responses over all of the links.

Second, an important dimension of the optimal biased contest design problem is missing in a two-player setting. With more than two contestants, setting biased rules not only manipulates the competitive balance of the playing field, but also serves to select preferred contestants: A designer can effectively exclude a contestant by imposing an excessively strong handicap on him, thereby diminishing his winning chances and discouraging him from active bidding; this is possible only if the contest involves at least three contenders. Which contestants should be excluded from the competition is an intriguing question, and can be explored only in a setting with three or more players.

A more general analysis would vastly expand the scope of the contest design problem, but imposes substantial complications. Optimal contest design results in a mathematical program with equilibrium constraints (MPEC) and typically requires an implicit programming approach. This approach requires that we solve for the equilibrium bidding strategies for any given parameterized contest rule, insert the equilibrium solution into the objective function, and solve for the optimal rule that maximizes the contest objective (e.g., Franke, Kanzow, Leininger and Schwartz, 2013). The approach loses its bite when more than two contestants are involved, as an asymmetric n-player contest, in general, yields no closed-form equilibrium solution. This limitation casts doubt on the generality of the conventional wisdom obtained in restricted settings. Our paper proposes an alternative optimization approach that allows us to characterize the optimum without solving explicitly for the equilibrium.

Next, we provide a snapshot of the approach and its underlying logic.

Optimization Approach

We adopt the frequently used framework of a generalized lottery contest to model a noisy contest in which higher effort does not ensure a win; this implies either a random underlying production process or an imprecise performance evaluation system. Imagine a winner-take-all contest with $n \geq 2$ players who differ in their prize valuations. For a given effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, a contestant wins with a probability

$$p_i(\boldsymbol{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)},$$

where $f_i(\cdot)$ maps one's effort outlays onto his effective output and is conventionally called the *impact function* of contestant $i \in \{1, \ldots, n\}$. We focus on the two most popularly adopted instruments for identity-dependent preferential treatments in the literature. The impact function takes the form

$$f_i(x_i; \alpha_i, \beta_i) = \alpha_i \cdot h(x_i) + \beta_i,$$

where α_i, a *multiplicative bias*, influences the marginal output of one's effort, while β_i, *an additive headstart*, directly adds to his output. The designer imposes treatment (α_i, β_i), with $\alpha_i, \beta_i \geq 0$, on each contestant i; the vector $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \equiv \big((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)\big)$ represents the prevailing contest rule and depicts how each contestant is favored or handicapped vis-à-vis his opponents.

The contest game is unsolvable in general. We propose an indirect approach to the optimization problem. Its primary logic is laid out as follows. Despite the lack of a solution to the equilibrium, a unique equilibrium is shown to exist for any feasible contest rule under mild regularity conditions. The equilibrium condition alludes to a correspondence between contestants' equilibrium winning probability distribution and their effort profile. The objective for contest design can be rewritten accordingly as a function of equilibrium winning probability distribution. Instead of optimizing over the choice of contest rule, the winning probability distribution is treated as a design variable: We let the designer directly assign winning probabilities among contestants to maximize the reformulated objective function. Finally, we demonstrate that any winning probability distribution can be induced by a contest rule in equilibrium, which closes the loop.

The key element of the approach is the aforementioned correspondence that enables us to reformulate the objective function. It provides a system of equations, and each expresses an individual's equilibrium effort as a function of his own *equilibrium winning odds* and *prize valuation*. Counterintuitively, neither the contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ nor opponents' equilibrium efforts appear in the function; the roles they play in the equilibrium are encapsulated in the contestant's equilibrium winning probability. The correspondence literally dissolves the linkage between contestants in the game, and disaggregates the strategic interaction into a series of individual decision problems. The optimization problem thus reduces to a simple programming that allocates probability mass among contestants based on their prize valuations.

In addition to the technical novelty, our approach unravels the nature of incentive provision in contests. When choosing his effort, a contestant is ultimately motivated by two factors: (i) the (*exogenous*) *reward* for his win, and (ii) the (*endogenous*) *prospect* for his win, i.e., his expectation of winning odds. Our approach—by assigning winning odds based on contestants' prize valuations—dismisses the distraction caused by the complex strategic interactions in the game, and directly internalizes the trade-offs in contestants' effort choice.

Implications and Applications

Our approach could substantially expand the scope of the analysis of optimal biased contests. In this paper, we set up a general objective function that addresses a wide spectrum of concerns in contest design; we primarily focus on the qualitative implications that reveal general properties of optimal biased contests.[4] The main results are summarized below.

Suboptimality of Headstart Allowing for additive headstarts $\boldsymbol{\beta}$—in addition to the freedom to set multiplicative biases $\boldsymbol{\alpha}$—cannot further improve the performance of the contest when the designer benefits from contestants' efforts. It is thus without loss of generality to focus solely on the optimal choice of multiplicative biases $\boldsymbol{\alpha}$.

[4] When a specific functional form is imposed on $h(\cdot)$—e.g., a Tullock contest with $h(x_i) = (x_i)^r$—a closed-form solution can be obtained for the optimum, which enables convenient and lucid analyses in parameterized settings.

A General Exclusion Principle The literature has debated whether certain players should be excluded from the competition (e.g., Baye, Kovenock and de Vries, 1993; and Fang, 2002). In contrast to previous studies that allow for outright exclusion, we consider implicit exclusion by setting identity-dependent biases. Under a general objective function that addresses broad concerns in contest design, we show that the optimal exclusion is monotone in the sense that exclusion always starts from the bottom-ranked contestants.

Maximizing Total Effort and the Expected Winner's Effort We apply our approach to the classical effort-maximizing problem. We demonstrate that to maximize total effort in an n-player contest, the optimum must involve at least three active contestants whenever a sufficient pool of potential contestants is available. As a result, a two-player setting is a knife-edge case, as it is suboptimal when the contest involves more than two players. Further, the optimum precludes the possibility of a "superstar," in the sense that an individual contestant's winning odds must be strictly less than $1/2$. We then proceed to the objective of maximizing the expected winner's effort, which is popularly studied in the auction literature but less often in the contest literature, partly due to the nonadditivity and nonlinearity of the objective function. In contrast to maximization of total effort, we show that the optimum keeps only the two top-ranked contestants active.

Leveling the Playing Field: Reexamined The literature on biased contests has centered on two fundamental questions: (i) Should the optimal contest equalize contestants' winning odds (leveling the playing field in terms of ex post equilibrium outcomes)? (ii) Should the contest rule favor weaker contestants vis-à-vis their stronger opponents (by leveling the playing field in terms of ex ante biased contest rules)? Our analysis systematically addresses these questions in a more general setting, and the results overturn the conventional wisdom.

First, we demonstrate that to maximize total effort, a designer favors the weaker contestant in a two-player contest such that she always equalizes contestants' winning odds, which generalizes the conventional wisdom obtained in Tullock contest settings—i.e., $h(x_i) = (x_i)^r$. However, a perfectly leveled playing field is an artifact of bilateral competition. With three or more contestants, the ex ante strongest player can be the least likely winner. Contestants' equilibrium winning probabilities can even be nonmonotone with respect to the rankings of their prize valuations.[5] We construct a parameterized setting and provide comparative statics that illustrate how the contest technology $h(\cdot)$ affects the equilibrium winning probability distribution in the optimum.

[5] In a standard lottery contest with $h(x_i) = x_i$, Franke, Kanzow, Leininger and Schwartz (2013) show in a numerical example that the optimal biased contest rule favors ex ante weaker contestants but does not fully level the playing field, in the sense that an ex ante stronger contestant wins with a larger probability.

Second, we demonstrate that the optimum may even further upset the balance of the contest by biasing the rule in favor of ex ante stronger contestants; the optimal biases can be nonmonotone, in the sense that an ex ante middle-ranked contestant receives the most favoritism. A comparative statics analysis is conducted in a Tullock contest setting to illustrate the underlying logic.

The rest of the paper proceeds as follows. Section 2 describes the contest model and introduces the design instruments and contest objectives. Section 3 sets up the contest design problem, develops a novel optimization approach, and characterizes optimal asymmetric contests. Section 4 discusses the conventional wisdom of leveling the playing field in detail, and Section 5 concludes. Proofs not in the main text are relegated to Appendix B.

2 Setup and Preliminaries

In this section, we present the fundamentals of the underlying contest game.

2.1 Generalized Lottery Contests

There are $n \geq 2$ risk-neutral contestants competing for a prize. The prize bears a value $v_i > 0$ for each contestant $i \in \mathcal{N} \equiv \{1, \ldots, n\}$, with $v_1 \geq \ldots \geq v_n > 0$, which is common knowledge. A contestant's prize valuation is a measure of his strength, as a higher valuation tends to incentivize more effort. To win the prize, contestants simultaneously submit their effort entries $x_i \geq 0$ and incur a cost of $c(x_i)$. The heterogeneity among contestants is encapsulated in the different prize valuations, which allows for an unambiguous ranking of contestants in terms of their strength and facilitates comparative static analyses.

We consider a generalized lottery contest with a ratio-form contest success function: For a given effort profile $\boldsymbol{x} \equiv (x_{1,}, \ldots, x_n)$, a contestant i wins the prize with a probability

$$p_i(\boldsymbol{x}) = \begin{cases} \dfrac{f_i(x_i)}{\sum_{j=1}^n f_j(x_j)} & \text{if } \sum_{j=1}^n f_j(x_j) > 0, \\ \dfrac{1}{n} & \text{if } \sum_{j=1}^n f_j(x_j) = 0, \end{cases} \tag{1}$$

where the function $f_i(\cdot)$, labeled the impact function in the contest literature, converts one's effort into his effective output in the lottery contest and satisfies $f_i(x_i) \geq 0$ for all $x_i \geq 0$. Obviously, a contestant $i \in \mathcal{N}$ is excluded from the contest if $f_i(x_i) = 0$ for all $x_i \geq 0$. In the extreme case in which one contestant has an increasing impact function, while every other contestant's impact function is a zero constant, we assume that he wins automatically.[6]

[6]This assumption is imposed to guarantee the existence of a pure-strategy Nash equilibrium.

Appendix A presents two rationales for the model's microeconomic underpinning: (i) a noisy-ranking approach adapted from the discrete-choice model (Clark and Riis, 1996; Jia, 2008); and (ii) a research tournament analogy (Loury, 1979; Dasgupta and Stiglitz, 1980; Fullerton and McAfee, 1999; Baye and Hoppe, 2003).

Given the effort profile $\boldsymbol{x} \equiv (x_1, \dots, x_n)$ and the above contest success function (1), contestant i's expected payoff can be written as

$$\pi_i(\boldsymbol{x}) := p_i(\boldsymbol{x}) \cdot v_i - c(x_i).$$

The set of impact functions $\{f_i(\cdot)\}_{i=1}^n$, together with contestants' valuations $\boldsymbol{v} \equiv (v_1, \dots, v_n)$ and the effort cost function $c(\cdot)$, defines a simultaneous-move contest game.

2.2 Regularity Condition and Equilibrium Property

We impose the following regularity condition on the contest game.

Definition 1 ***(Regular Concave Contests)*** *A contest $\left(\boldsymbol{v}, \{f_i(\cdot)\}_{i=1}^n, c(\cdot)\right)$ is called a regular concave contest if (i) the impact function for contestant $i \in \mathcal{N}$ is either a nonnegative constant or a twice-differentiable function, with $f_i(x_i) \geq 0$, $f_i'(x_i) > 0$, and $f_i''(x_i) \leq 0$ for all $x_i \geq 0$; and (ii) the effort cost function satisfies $c(0) = 0$, $c'(x_i) > 0$ and $c''(x_i) \geq 0$ for all $x_i > 0$.*

The above definition simply requires the usual concave impact functions and a convex effort cost function. These regularity conditions ensure that a contestant's payoff function is concave in effort, and is widely adopted in the literature. Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) prove the existence and uniqueness of the equilibrium in the above contest game under the restriction of $f_i(0) = 0$ for all $i \in \mathcal{N}$. Therefore, their results cannot be applied directly to contests in which headstarts are in place, i.e., $f_i(0) > 0$ for some $i \in \mathcal{N}$. The following theorem generalizes their results by relaxing the zero-headstart assumption.

Theorem 1 ***(Existence and Uniqueness of Equilibrium)*** *There exists a unique pure-strategy Nash equilibrium in a regular concave contest game $\left(\boldsymbol{v}, \{f_i(\cdot)\}_{i=1}^n, c(\cdot)\right)$.*

Our study focuses on the above-defined concave contests for two reasons. First, when impact functions are convex, a pure-strategy equilibrium tends to fade away. Although mixed-strategy equilibria exist, they generally are not unique and their specific properties remain elusive in the literature (e.g., Ewerhart, 2015, 2017). Second, the regularity condition implies a production technology with nonincreasing marginal output, which is common in many real-life scenarios. This condition can also be understood in a natural and intuitive manner when the contest model is interpreted as the aforementioned research tournament à la Fullerton and McAfee (1999), in which case diminishing marginal return can naturally be expected: In scientific research, duplicating input increases the likelihood of success, but does not necessarily double the chance of a discovery.[7]

2.3 Design Instruments and Contest Objectives

Theorem 1 ensures the existence and uniqueness of a pure-strategy equilibrium in the underlying contest game, which allows us to set up the contest design problem in a two-stage structure. In the first stage, anticipating contestants' strategic plays in the second stage, the designer sets the contest rule and announces it publicly. In the second stage, contestants exert effort simultaneously to vie for the prize. We first discuss the instruments available to the designer and then elaborate on the properties and implications of the objective function.

2.3.1 Design Instruments

As mentioned previously, we follow the tradition in the literature and mainly focus on two types of instruments to model identity-dependent preferential treatment: (i) multiplicative biases—i.e., weights on contestants' effective output—and (ii) additive headstarts. To put this formally, the impact function takes the form

$$f_i\left(x_i; \alpha_i, \beta_i\right) = \alpha_i \cdot h\left(x_i\right) + \beta_i. \tag{2}$$

The function $h(\cdot)$ is exogenously given as the fundamental *contest technology*; the identity-dependent treatment imposed on each contestant $i \in \mathcal{N}$ is given by a tuple (α_i, β_i), with $\alpha_i, \beta_i \geq 0$.[8] The contest technology $h\left(x_i\right)$ is assumed to have the following properties.

Assumption 1 ***(Concave Contest Technology)*** *$h(\cdot)$ is twice differentiable, with $h'(x) > 0$, $h''(x) \leq 0$, and $h(0) = 0$.*[9]

Both the multiplicative bias, α_i, and the additive headstart, β_i, are popularly adopted in the literature to model preferential treatments. Fu (2006); Franke (2012); Franke, Kanzow, Leininger and Schwartz (2013, 2014); and Epstein, Mealem and Nitzan (2011) focus on the

[7]See Appendix A for further discussion.

[8] Drugov and Ryvkin (2017) study a two-player contest in which contestant 1 wins with a probability $p_1 = (x_1 + \beta)/(x_1 + x_2)$, and contestant 2 wins with a probability $1 - p_1$. Their setting is equivalent to one in which contestants 1 and 2 are imposed with an identity-dependent headstart of β and $-\beta$, respectively. The model differs from ours in that we assume $\beta_i \geq 0$ for all $i \in \mathcal{N}$.

[9]With $\alpha_i, \beta_i \geq 0$, Assumption 1 ensures that the game satisfies the requirements of Definition 1, and Theorem 1 applies, by which a unique pure-strategy equilibrium exists.

former, while Clark and Riis (2000); Konrad (2002); Siegel (2009, 2014); Kirkegaard (2012); and Li and Yu (2012) consider the latter. Franke, Leininger and Wasser (2018) allow for both. Both instruments vary a contestant's (deterministic) output, but through starkly different channels: α_i scales up or down a contestant's output for any given effort, while β_i directly boosts it regardless of his effort. The contrast inspires interesting comparisons, which generate useful implications for contest design.

We assume for convenience that all contestants are endowed with the same contest technology $h(\cdot)$, and thus contestants' heterogeneity is encapsulated in the difference in their prize valuations. This setting enables lucid comparative statics. It is noteworthy that our analysis can readily be extended to a setting that allows for heterogeneous contest technologies $\{h_i(\cdot)\}_{i=1}^n$, which will be discussed in Section 5.

It is useful to point out that our analysis primarily focuses on moral hazard situations in which economic agents' efforts are hidden choices and difficult to contract upon (see, for instance, Lazear and Rosen, 1981, and Gershkov, Li and Schweinzer, 2009). The designer assigns the treatment in anticipation of contestants' strategic choice of efforts, but implementing these instruments does not require that efforts be observable or verifiable. Consider the aforementioned example of CEO succession races: A key appointment improves the favorite candidate's productivity, while his/her actual effort is hardly verifiable.

2.3.2 A General Objective Function

Prior to the competition, the designer chooses $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to maximize an objective function $\Lambda(\cdot)$, which is a function of the effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$; the profile of winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$; and the profile of prize valuations $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. We impose the following regularity condition on the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$.

Assumption 2 ***(Objective Function)*** *Fixing $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ and $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$, $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ is weakly increasing in x_i for all $i \in \mathcal{N}$.*

The assumption simply requires that contestants' efforts accrue to the benefit of the contest designer: For a given winning probability distribution $\boldsymbol{p}$, an increase in a contestant's effort does not reduce her payoff.

The objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ encompasses a wide array of scenarios. Let us first consider the following:

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \sum_{i=1}^{n} x_i + \psi \sum_{i=1}^{n} p_i v_i - \gamma \sum_{i=1}^{n} \left(p_i - \frac{\sum_{j=1}^{n} p_j}{n} \right)^2, \tag{3}$$

with $\psi \geq 0$ and $\gamma \geq 0$. The function obviously satisfies the requirement of Assumption 2.

When the weights ψ and γ both reduce to zero, the above expression boils down to $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} x_i$, the popularly studied objective of total effort maximization. The objective function (3) allows the designer to have a direct preference for contestants' winning probability distribution. The term $\sum_{i=1}^{n} \left(p_i - (\sum_{j=1}^{n} p_j)/n\right)^2$ is the variance of the winning probabilities. With $\gamma > 0$, the designer prefers a less predictable outcome. For instance, in sports competitions, spectators often not only appreciate contenders' efforts, but also demand more suspense about the eventual winner (see Chan, Courty and Hao, 2008; and Ely, Frankel and Kamenica, 2015).[10] The contest objective also accommodates the pursuit of selection efficiency (see Meyer, 1991; Hvide and Kristiansen, 2003; Ryvkin and Ortmann, 2008; and Fang and Noe, 2018): The additional component $\sum_{i=1}^{n} p_i v_i$ strictly increases when a contestant of a higher valuation is able to win more often, which also provides an example of how contestants' prize valuations could directly affect the designer's payoff.[11]

In many competitive events, however, only the winner's performance is relevant to the organizer's interest. Suppose that the contest designer does not care about the overall effort, but only the expected winner's effort. The objective function can be written as

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} p_i x_i. \tag{4}$$

This contest objective has gained increasing attention in the literature (e.g., Moldovanu and Sela, 2006; Serena, 2017; and Barbieri and Serena, 2019) Imagine, for instance, that the buyer in a procurement tournament is only concerned about the quality of the winning product; a similar observation can be seen in an architectural design competition. Also, a CEO succession race motivates candidates to develop their managerial skills: Large public firms—e.g., GE and HP—often have difficulty retaining losing candidates, which would lead them to focus only on the acquisition of human capital from the winner (Fu and Wu, 2019b). Design objective (4) clearly satisfies Assumption 2.

[10] Such a preference is also assumed by Fort and Quirk (1995), Szymanski (2003), and Runkel (2006) in two-player settings.

[11] The contest designer may care about both effort supply and contestants' welfare (e.g., Epstein, Mealem and Nitzan, 2011). Recall that a contestant i has an expected payoff $\pi_i = p_i v_i - x_i$ with linear effort cost functions. This preference can formally be expressed as $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) := \phi \sum_{i=1}^{n} \pi_i + (1 - \phi) \sum_{i=1}^{n} x_i = \phi \sum_{i=1}^{n} p_i v_i + (1 - 2\phi) \sum_{i=1}^{n} x_i$. Assumption 2 is satisfied if and only if $\phi \leq \frac{1}{2}$, in which case this objective function boils down to a case of the objective function (3). Higher efforts, however, would cause net disutility to the designer if her preference over contestants' welfare is excessively strong—i.e., $\phi > \frac{1}{2}$—which defies Assumption 2.

3 Optimal Contest Design: Analysis

Given the existence and uniqueness of a pure-strategy equilibrium in the contest game for arbitrary $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, the optimal contest design problem yields a typical mathematical program with equilibrium constraints (MPEC): Contestants' equilibrium effort profile, $\boldsymbol{x}$, is endogenously determined in the equilibrium as a function of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ set by the designer, and the designer chooses $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for the following optimization problem:

$$
\begin{aligned}
& \max_{\{\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta}\}} \Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) \\
\text{subject to } x_i \quad &= \quad \arg\max_{x_i \geq 0} \pi_i(\boldsymbol{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}), \\
p_i(\boldsymbol{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) \quad &= \quad \begin{cases} \dfrac{f_i\left(x_i; \alpha_i, \beta_i\right)}{\sum_{j=1}^n f_j(x_j; \alpha_j, \beta_j)} & \text{if } \sum_{j=1}^n f_j(x_j; \alpha_j, \beta_j) > 0, \\ \dfrac{1}{n} & \text{if } \sum_{j=1}^n f_j(x_j; \alpha_j, \beta_j) = 0. \end{cases}
\end{aligned}
$$

The conventional approach of solving the MPEC requires an equilibrium solution of effort profile $\boldsymbol{x}$ for an arbitrary $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, which is, in general, unavailable. This nullifies the conventional approach and calls for an alternative technique. We take a detour to bypass the difficulty, and the approach can be described as follows:

i. We resort to the first-order conditions for the unique equilibrium of a contest game under an arbitrary contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and show that the optimum can always be achieved by a contest rule with zero headstart. This allows us to simplify the optimization problem.

ii. According to the simplified optimization problem, we establish a correspondence between contestants' equilibrium effort profile $\boldsymbol{x}$ and equilibrium winning probability distribution $\boldsymbol{p}$.

iii. Based on the correspondence mentioned above, we rewrite the objective as a function of the winning probability distribution. Instead of searching directly for the optimal contest rule, we let the designer assign equilibrium winning probabilities to contestants. We then solve for the probability distribution that maximizes the objective function.

iv. Finally, we identify the contest rule that induces the desirable winning probability distribution in equilibrium.

In the rest of this section, we first lay out the fundamentals of our approach and then apply it to optimal contest design problems.

In the unique equilibrium of a contest game, the first-order condition $\partial\pi_i(\boldsymbol{x})/\partial x_i = 0$ must be satisfied for an active contestant $i \in \mathcal{N}$. With the impact functions specified in expression (2), the condition can be rewritten as

$$\frac{\sum_{j\neq i}\left[\alpha_j h(x_j)+\beta_j\right]}{\left\{\sum_{j=1}^{n}\left[\alpha_j h(x_j)+\beta_j\right]\right\}^2}\cdot h'(x_i) = \frac{1}{\alpha_i v_i}\cdot c'(x_i), \text{ for } x_i > 0.$$

Similarly, the following inequality holds if contestant i remains inactive in equilibrium:

$$\frac{\sum_{j\neq i}\left[\alpha_j h(x_j)+\beta_j\right]}{\left\{\sum_{j=1}^{n}\left[\alpha_j h(x_j)+\beta_j\right]\right\}^2}\cdot h'(x_i) \leq \frac{1}{\alpha_i v_i}\cdot c'(x_i), \text{ for } x_i = 0.$$

The above equilibrium conditions, together with the winning probability $p_i(\boldsymbol{x})$ specified in Equation (1), imply immediately that

$$p_i(1-p_i)v_i = c'(x_i)\cdot\frac{\alpha_i h(x_i)+\beta_i}{\alpha_i h'(x_i)}, \text{ for } x_i > 0,\text{[12]} \tag{5}$$

and

$$p_i(1-p_i)v_i \leq c'(x_i)\cdot\frac{\alpha_i h(x_i)+\beta_i}{\alpha_i h'(x_i)}, \text{ for } x_i = 0. \tag{6}$$

Conditions (5) and (6) establish the relationship between efforts and winning probabilities in equilibrium: The left-hand side is a quadratic function of the equilibrium winning probability p_i, while the right-hand side is a strictly increasing function with respect to equilibrium effort x_i. This relationship lays a foundation for our subsequent analysis, and many general insights ensue.

3.1 Suboptimality of Additive Headstart

Equilibrium conditions (5) and (6) allow us to compare the two design instruments: multiplicative biases vs. additive headstarts. We now demonstrate that the former outperforms the latter. To put this more specifically, we show that fixing an arbitrary contest rule with positive headstarts, we can always construct an alternative contest rule with zero headstart that induces the same equilibrium winning probability distribution but strictly higher effort. As a result, the alternative contest rule generates a weakly higher payoff to the designer under Assumption 2, which in turn indicates that the designer cannot strictly benefit from her freedom to award additive headstart.

[12] We need $\alpha_i > 0$ for the right-hand side to be well defined, which clearly holds. In fact, if $\alpha_i = 0$, it is straightforward to see that $x_i = 0$ is a strictly dominant strategy for player i due to the fact that costly effort has zero impact on player i's winning probability.

A sketch proof is provided below to unveil the key logic for the result. Denote by $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) \equiv \big((\alpha_1^*, \ldots, \alpha_n^*), (\beta_1^*, \ldots, \beta_n^*)\big)$ the optimal contest rule that maximizes $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$; the corresponding equilibrium effort profile and winning probabilities are denoted by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_n^*)$ and $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$, respectively. Suppose that $\beta_t^* > 0$ for some $t \in \mathcal{N}$ in the optimum. Let us focus on the case of an active contestant t, i.e., $x_t^* > 0$, as the logic naturally extends to inactive ones with $x_t^* = 0$.

Recall the equilibrium condition

$$p_t^*(1 - p_t^*)v_t = c'(x_t^*) \cdot \frac{\alpha_t^* h(x_t^*) + \beta_t^*}{\alpha_t^* h'(x_t^*)}.$$

Denote by $x^\dagger$ the unique solution to the following equation:

$$c'(x_t^*) \cdot \frac{\alpha_t^* h(x_t^*) + \beta_t^*}{\alpha_t^* h'(x_t^*)} = c'(x^\dagger) \cdot \frac{h(x^\dagger)}{h'(x^\dagger)}.^{13} \tag{7}$$

Simple analysis would verify that $x^\dagger > x_t^*$, given $\beta_t^* > 0$. Consider an alternative contest rule with $\widetilde{\boldsymbol{\alpha}} \equiv (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)$ and $\widetilde{\boldsymbol{\beta}} \equiv (\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n)$, such that

$$\left(\widetilde{\alpha}_i, \widetilde{\beta}_i\right) := \begin{cases} \left(\frac{\alpha_t^* h(x_t^*) + \beta_t^*}{h(x^\dagger)}, 0\right) & \text{for} \quad i = t, \\ (\alpha_i^*, \beta_i^*) & \text{for} \quad i \neq t. \end{cases}$$

In words, all contestants are awarded the same identity-dependent treatment as before except for contestant t. Under this new contest rule, headstart is removed for contestant t. Simple algebra verifies that the equilibrium effort profile under the new contest rule $(\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}})$—which we denote by $\widetilde{\boldsymbol{x}}^* \equiv (\widetilde{x}_1^*, \ldots, \widetilde{x}_n^*)$—is given by

$$\widetilde{x}_i^* = \begin{cases} x^\dagger & \text{for} \quad i = t, \\ x_i^* & \text{for} \quad i \neq t. \end{cases}$$

The new contest rule outperforms under Assumption 2. It induces the same winning probability distribution, because $\widetilde{\alpha}_t \cdot h(x^\dagger) + \widetilde{\beta}_t = \alpha_t^* \cdot h(x_t^*) + \beta_t^*$ by our construction, while the effort of contestant t strictly increases because $x^\dagger > x_t^*$ by Equation (7).[14] This argument leads to the following.

[13] The existence and uniqueness of the solution $x^\dagger$ follows from the facts that $c'(x) \cdot h(x)/h'(x)$ is strictly increasing in x, $\lim_{x \searrow 0} c'(x) \cdot h(x)/h'(x) = 0$, and $\lim_{x \nearrow \infty} c'(x) \cdot h(x)/h'(x) = \infty$.

[14] A closer inspection of Equation (7) indicates that $x^\dagger > x_t^*$ may not hold if the headstart β_t is allowed to be negative, in which case the comparison depends on the properties of $c'(\cdot)$, $h(\cdot)$, and $h'(\cdot)$. Drugov and Ryvkin (2017) allow for negative headstart (see Footnote 8) and show that headstart can be optimal, depending on the sign of $c'''(\cdot)$.

Theorem 2 ***(Suboptimality of Headstart)*** *Suppose that Assumptions 1 and 2 are satisfied. The optimum can always be achieved by choosing multiplicative biases* $\boldsymbol{\alpha}$ *only and setting headstarts* $\boldsymbol{\beta}$ *to zero.*

It is thus without loss of generality to abstract away headstart and focus on multiplicative biases when searching for the optimal biased contests, i.e., assuming $f_i(x_i; \alpha_i, \beta_i) = \alpha_i \cdot h(x_i)$, with $\beta_i = 0$ for all $i \in \mathcal{N}$.[15] Franke, Leininger and Wasser (2018, Proposition 3.6) obtain similar results. Specifically, they show in a standard lottery contest—i.e., $h(x_i) = x_i$—that a positive headstart is suboptimal when the designer aims to maximize total effort. Our analysis generalizes Franke et al. (2018) in two dimensions: First, we allow for a flexible contest technology, and second, the optimization problem addresses a broad objective.[16]

3.2 Reformulated Design Problem

Theorem 2 allows us to derive the fundamental equilibrium correspondence that underpins our optimization approach: With $\beta_i = 0$, the following must hold in an equilibrium:

$$p_i(1 - p_i)v_i = c'(x_i) \cdot \frac{h(x_i)}{h'(x_i)}, \forall\, i \in \mathcal{N}. \tag{8}$$

A system of n set-valued functional equations depicts the relation between winning probability distribution $\boldsymbol{p}$ and contestants' effort profile $\boldsymbol{x}$ in equilibrium, with the right-hand side strictly increasing with x_i. For convenience, we call the system of equations the *equilibrium correspondence* of the contest game. The correspondence reminds us of the first-order condition (5) for an active player. However, it also holds for an *inactive* contestant, as $x_i = 0$ is associated with $p_i = 0$. Further, define the inverse of $\log(c'(x) \cdot h(x)/h'(x))$ by $g(\cdot)$.[17] The correspondence (8) can be rewritten as

$$x_i = g\left(\log(p_i\left(1 - p_i\right)) + \log(v_i)\right), \forall\, i \in \mathcal{N}. \tag{9}$$

Two remarks are in order before we proceed. First, each equation in the system of equations (9) literally delineates a direct and unique relation between x_i and (p_i, v_i) for an individual contestant $i \in \mathcal{N}$. The equilibrium probability p_i can be viewed as a *sufficient*

[15] Headstarts, however, can be preferred to multiplicative biases by an effort-maximizing contest designer in all-pay auctions. See Li and Yu (2012) and Franke, Leininger and Wasser (2018) for more details.

[16] Allowing for a general objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ substantially enriches the optimization problem. Adjusting design instruments varies both $\boldsymbol{x}$ and $\boldsymbol{p}$, which obscures the optimum. In proving Theorem 2, our analysis bypasses the difficulty by constructing an alternative contest rule that preserves the same winning probability distribution while boosting efforts.

[17] Assumption 1 and the convexity of the effort cost function imply that $g(\cdot)$ is well defined. In particular, $g(\cdot)$ is a strictly increasing function, with $g(-\infty) = 0$ and $g(\infty) = \infty$.

statistic of the equilibrium in the contest: p_i is not exogenously given, but a function endogenously determined jointly by contestants' equilibrium effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ and the treatment $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$. Second, the correspondence (9) unveils the nature of incentive provision in contests. A contestant's effort decision, regardless of the game theoretical structure of the contest, takes into account two basic factors: (i) value (v_i), i.e., how much he can be rewarded when he wins; and (ii) prospect (p_i), i.e., the expectation about how likely he is to win.

The correspondence (9) opens a new avenue for contest design. The objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ can be rewritten as $\Lambda\big(\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v}), \boldsymbol{p}, \boldsymbol{v}\big)$; instead of setting $\boldsymbol{\alpha}$ directly, we treat winning probability distribution $\boldsymbol{p}$ as the design variable and let the designer maximize $\Lambda\big(\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v}), \boldsymbol{p}, \boldsymbol{v}\big)$, subject to (9) and the following feasibility constraints:

$$\sum_{i=1}^{n} p_i = 1, \text{ and } p_i \geq 0, \text{ for all } i \in \mathcal{N}. \tag{10}$$

A maximizer automatically exists for any smooth and continuous objective $\Lambda\big(\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v}), \boldsymbol{p}, \boldsymbol{v}\big)$ given that the choice set, defined by (10), is an $(n-1)$-dimensional simplex.

The following result is established as the last piece of the puzzle.

Theorem 3 ***(Implementing Winning Probabilities by Setting Biases)*** *Fix any equilibrium winning probability distribution $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$.*

i. If $p_j = 1$ for some $j \in \mathcal{N}$, then $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can be induced by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

ii. If there exist at least two active contestants, then $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can be induced by the following set of biases $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \dfrac{p_i}{h\Big(g\big(\log(p_i(1-p_i)) + \log(v_i)\big)\Big)} & \text{if} \quad p_i > 0, \\ 0 & \text{if} \quad p_i = 0. \end{cases} \tag{11}$$

Theorem 3 formally states that the contest designer can properly construct the set of weights $\boldsymbol{\alpha}$ to induce any equilibrium winning probability distribution.[18] The result closes

[18]It should be noted that the biases $\boldsymbol{\alpha}$ that induce each given $\boldsymbol{p}$ are not unique. For instance, the same equilibrium outcome can be induced by multiplying all α_i by some positive factor.

the loop for the reformulated optimization problem: Upon obtaining the maximizer to $\Lambda\big(\boldsymbol{x}(\boldsymbol{p},\boldsymbol{v}),\boldsymbol{p},\boldsymbol{v}\big)$, the optimal biases $\boldsymbol{\alpha}^* \equiv (\alpha_1^*,\ldots,\alpha_n^*)$ can readily be obtained by invoking Theorem 3.

Consider, for example, the widely studied Tullock contest with $h(x_i) = (x_i)^r$ and assume a linear effort cost function $c(x_i) = x_i$. An equation in the correspondence (9) boils down to

$$x_i = rp_i(1-p_i)v_i.$$

The above-mentioned objective function (3) can be rewritten as

$$\Lambda\big(\boldsymbol{x}(\boldsymbol{p},\boldsymbol{v}),\boldsymbol{p},\boldsymbol{v}\big) := \sum_{i=1}^{n}\big[rp_i(1-p_i)v_i\big] + \psi\sum_{i=1}^{n} p_i v_i - \gamma\sum_{i=1}^{n}\left(p_i - \frac{\sum_{j=1}^{n} p_j}{n}\right)^2,$$

which gives rise to a quadratic programming. Standard technique would obtain a handy closed-form solution to the optimal biases $\boldsymbol{\alpha}$.[19] Our paper, however, would primarily focus on the general implications of the contest design problem, instead of solving for closed-form solutions in specific settings.

The reformulation enormously simplifies the design problem. By the equilibrium correspondence (9), each contestant chooses his effort as if he responds merely to (p_i, v_i), his own winning odds and prize valuation: The strategic linkages between contestants seemingly dissolve when the winning probability distribution is treated as a design variable. This approach insulates the designer from the distraction of the complex strategic interaction of the contest game; instead, the reformulated optimization problem boils down to a simple programming that allocates probability mass among contestants purely based on the profile of their prize valuations.

3.3 A General Exclusion Principle

We now derive a general property of the optimal contest based on our approach. Recall that the contest designer, when setting $\boldsymbol{\alpha}$, can effectively exclude a contestant by imposing zero weight on his entry, which discourages him from exerting positive effort. The design problem involves a hidden dimension: the selection of active contestants in the competition. In other words, which contestants should be included in the optimal contest?

Define $\tau : \mathcal{N} \to \mathcal{N}$ as a permutation of the set of players $\mathcal{N} \equiv \{1,\ldots,n\}$. In particular, player i is replaced by player $\tau(i)$ in the rearrangement. With slight abuse of notation, let us define $\tau(\boldsymbol{x}) := (x_{\tau(1)},\ldots,x_{\tau(n)})$, $\tau(\boldsymbol{p}) := (p_{\tau(1)},\ldots,p_{\tau(n)})$, and $\tau(\boldsymbol{v}) := (v_{\tau(1)},\ldots,v_{\tau(n)})$.

[19] The application of our optimization approach and the solutions to optimal biases in Tullock contest settings are available from authors upon request.

Similarly, let $\tau_{ij}(\boldsymbol{x})$ denote the permutation obtained by swapping contestants i and j.

To obtain more mileage, we impose the following condition on the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$.

Assumption 3 *The contest designer's objective $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ satisfies the following properties:*

i. for all permutations τ of $\mathcal{N}$, $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \Lambda\big(\tau(\boldsymbol{x}), \tau(\boldsymbol{p}), \tau(\boldsymbol{v})\big)$;

ii. if $(p_i, x_i) = (0, 0)$ for some contestant $i \in \mathcal{N}$, then $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) \leq \Lambda\big(\boldsymbol{x}, \boldsymbol{p}, \tau_{ij}(\boldsymbol{v})\big)$ for all $j \in \mathcal{N}$ such that $v_j < v_i$;

iii. fixing $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ and $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$, $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ is strictly increasing in x_i if $p_i > 0$.

Part (i) of the above assumption implies that the designer's preference is anonymous: She does not have ex ante preference over certain players. Our setting allows the designer to directly benefit from the values contestants derived from the prize, i.e., $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. Part (ii) of the assumption indicates that the prize value for a contestant is more likely to accrue to the designer's benefit when he is active. The requirement is automatically satisfied in the simplest case in which the objective function is independent of contestants' prize valuations, e.g., in which the designer maximizes total effort or the expected winner's effort. Part (iii) states that the designer would strictly benefit if an active player exerts more effort.

Part (iii) of Assumption 3 immediately implies Assumption 2:[20] Theorem 2 thus remains in place, and headstarts are suboptimal for contest design under Assumption 3. Despite the additional requirements, Assumption 3 is by no means restrictive. It is straightforward to verify that all of the examples discussed in Section 2.3.2 satisfy the requirements. We obtain the following.

Theorem 4 ***(Exclusion Principle)*** *Suppose that Assumptions 1 and 3 are satisfied. If $p_i^* = 0$ for some $i \in \mathcal{N}$ in the optimum, then $p_j^* = 0$ for all $j \in \mathcal{N}$, with $v_j < v_i$.*

By Theorem 4, exclusion in the optimum must be *monotone*. That is, whenever the designer intends to exclude contestants, she must target the ex ante weakest. This result stands in contrast to those obtained in previous studies. In an all-pay auction, Baye, Kovenock and de Vries (1993) show that an effort-maximizing contest designer may strategically exclude the strongest contestant. In contrast, Fang (2002) demonstrates that the designer does not have a strict incentive to exclude players from a lottery contest—i.e., $h(x_i) = x_i$—and thus the exclusion principle of the all-pay auction identified by Baye et al. (1993) does not extend

[20] To be more rigorous, we need to impose the condition that $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v})$ is weakly increasing in x_i at $p_i = 0$ for all $i \in \mathcal{N}$.

to lottery contests. Both studies assume total effort maximization and outright exclusion, while we allow for a general objective function and an indirect exclusion approach, i.e., assigning zero or excessively small weights to discourage participation. Theorem 4 provides a new exclusion principle when the designer is able to bias the contest, and complements the literature on player exclusion.

The monotone exclusion principle may compel one to conjecture that an ex ante stronger contestant—i.e., one with a larger v_i—would win with a (weakly) higher probability in the optimum. However, this does not necessarily hold in general. As will be discussed in depth in Section 4, the ranking of active contestants' equilibrium winning probabilities depends on a number of factors.

3.4 Optimal Contests: Maximizing Total Effort and the Expected Winner's Effort

To demonstrate the utility and versatility of our approach, we investigate two typical scenarios for contest design. First, we set ψ and γ in the objective function (3) to zero, and consider the situation in which the contest designer aims to maximize aggregate effort, i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} x_i$. Second, we consider the objective function (4), the maximization of the expected winner's effort—i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} p_i x_i$.

Maximizing Total Effort With slight abuse of notation, let us denote the effort-maximizing winning probabilities and the corresponding optimal biases by $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ and $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$, respectively. Consider a two-player contest with $v_1 \geq v_2$. It is well known in the literature that in a Tullock contest setting—i.e., $h(x_i) = (x_i)^r$—the optimum fully balances the playing field, with $p_1^* = p_2^* = \frac{1}{2}$, for all $r \in (0, 1]$. This can be achieved by setting α_2^* to $(v_1/v_2)^r$ with $(v_1/v_2)^r \geq 1$ and normalizing α_1^* to one.

A closer look at the aforementioned equilibrium correspondence reveals that this leveling-the-playing-field principle immediately extends regardless of the contest technologies. To see this, recall that in the equilibrium,

$$x_i = g\Big(\log\big(p_i(1-p_i)\big) + \log(v_i)\Big), \forall\, i \in \mathcal{N},$$

which indicates that x_i strictly increases with $p_i(1-p_i)$. Note that $p_i\,(1-p_i)$ is nonmonotone in p_i: It increases first and then drops, being maximized uniquely at $p_i = \frac{1}{2}$. To put this intuitively, one gives up when he faces a slim chance of winning, while he also slackens off when he expects an easy win, which underpins the nonmonotone best-response function in a standard contest game (Dixit, 1987). With a simple additive objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} x_i$, the following naturally emerges in the optimum without further analysis.

Theorem 5 ***(Winning Probabilities in Two-player Effort-maximizing Contests)*** *Suppose that $n = 2$, Assumption 1 is satisfied, and the designer aims to maximize total effort. Then the optimal contest perfectly levels the playing field, i.e., $p_1^* = p_2^* = 1/2$.*

When more than two players are involved in a contest, bias setting also determines the set of active contestants in the competition. Theorem 4 shows that any exclusion must be bottom-up, while the next result concerns itself with the proper number of active contestants and the equilibrium winning probability distribution in the optimum.

Theorem 6 ***(Effort-maximizing Contests with Three or More Players)*** *Suppose that $n \geq 3$, Assumption 1 is satisfied, and the designer aims to maximize total effort. Then the following statements hold:*

i. The optimal contest allows for at least three active players.

ii. The optimal contest does not allow any contestant to win with a probability more than $1/2$, i.e., $p_i^* < 1/2$, $\forall\, i \in \mathcal{N}$.

The first part of Theorem 6 generalizes Franke, Kanzow, Leininger and Schwartz (2013, Theorem 4.6), and shows that a head-to-head competition is suboptimal whenever a third contestant is available, regardless of the distribution of contestants' prize valuations. Although the claim does not seem obvious, the same correspondence would unravel it immediately. Suppose otherwise that in a multiplayer contest only two players are kept active. Optimization requires that they have equal chance to win. Recall that x_i strictly increases with $p_i(1 - p_i)$, and $p_i(1 - p_i)$ is maximized when $p_i = \frac{1}{2}$, with $d\left[p_i(1 - p_i)\right]/dp_i\big|_{p_i=1/2} = 0$. With a simple additive objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} x_i$, the designer can be strictly better off by adjusting contest rule $\boldsymbol{\alpha}$ to award a third player a very small probability of winning: In the new equilibrium, the third player contributes positive effort; the other two would barely reduce their efforts, because the marginal effect on $p_i(1 - p_i)$ is negligible. To summarize, allowing for a third player always boosts the performance of the contest regardless of his relative competence.

The second part of the theorem provides a key property of the optimum regarding the winning probability distribution. The optimum precludes a "superstar," in the sense that an individual contestant's winning odds must be strictly less than the sum of the others', i.e., $p_i^* < 1/2$, $\forall\, i \in \mathcal{N}$. First, it is never optimal to let contestant i win with a probability p_i strictly more than $1/2$. Otherwise, the designer can induce the same amount of effort from contestant i by assigning $1 - p_i$ instead and induce more effort from other contestants by allocating to them the saved probability mass $2p_i - 1$.

Second, assigning an equilibrium winning probability $1/2$ to contestant i is suboptimal. An argument analogous to that of Theorem 6(i) would unravel the nuance. Suppose, to the contrary, that $p_i = 1/2$. Then total effort must increase if a small probability mass is shifted away to another contestant j: $p_i(1-p_i)$ is concave and maximized at $p_1 = 1/2$, so the negative effect on contestant i is zero on the margin, while the corresponding positive effect on j is not.

The presence of an overwhelmingly dominant contestant stifles other contestants' incentive. However, it is unclear, in the case of $n \geq 3$, whether the optimal contest completely levels the playing field—i.e., $p_i^* = 1/n$—and whether an ex ante stronger contestant would necessarily be handicapped more, i.e., a larger v_i is associated with a smaller α_i in the optimum. We apply our approach to explore these classical questions in Section 4 and show that the conventional wisdom does not universally hold.

Maximizing the Expected Winner's Effort Next, we consider the design objective of maximizing the expected winner's effort. Unlike the maximization of aggregate effort $\sum_{i=1}^{n} x_i$, the objective function $\sum_{i=1}^{n} p_i x_i$ is nonadditive in the contestant's effort, because winning probabilities p_is are functions of effort profile $\boldsymbol{x}$ and are factored in multiplicatively, which complicates the analysis. Our approach is immune to the nuance because of the reformulation. Denote by $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ the winning probabilities in the optimal contest. We obtain the following.

Theorem 7 ***(Optimal Contest that Maximizes the Expected Winner's Effort)*** *Suppose that Assumption 1 is satisfied and the designer aims to maximize the expected winner's effort. Then only the two ex ante strongest contestants would remain active in the optimal contest. Moreover, the ex ante stronger player always wins with a strictly higher probability than the underdog, independent of the shape of $g(\cdot)$. That is, if $v_1 > v_2$, then $p_1^{**} > p_2^{**} > 0$.*[21]

By Theorem 7, the optimal contest that maximizes the expected winner's effort involves only two active contestants: All contestants other than the two ex ante strongest are excluded from the competition, which stands in contrast to the optimum established in Theorem 6 under total effort maximization when $n \geq 3$. Further, the playing field is never fully balanced, and the top dog always wins more often. This stands in contrast to the principle obtained in Theorem 5 for two-player effort-maximizing contests.

The result can again be interpreted in light of the correspondence (9). It is intuitive to infer that the optimum—which maximizes the weighted sum $\sum_{i=1}^{n} p_i x_i$—must concentrate the probability mass on the minimal number of the most productive contestants, i.e., the two strongest contestants. Further, suppose otherwise that the two active contestants win with equal chance. The designer can be strictly better off by shifting a small amount of

[21] It is straightforward to show that $p_1^{**} = p_2^{**} = 1/2$ if $v_1 = v_2$.

probability mass from p_2 to p_1. Recall that $x_i = g\Big(\log\big(p_i(1-p_i)\big) + \log(v_i)\Big)$. Its impact on $p_i(1-p_i)$ fades away on the margin, while a larger probability is attached to a higher effort: $x_1 > x_2$ because $v_1 > v_2$. In short, the optimal contest must sufficiently preserve individual incentives by limiting the number of contestants, and requires that the winning probability assignment be "assortative," i.e., a more productive contestant wins more often.

4 Leveling the Playing Field: Reexamined

In this section, we apply our approach to explore a classical question in the contest literature: How should the balance of the playing field be optimally set to maximize total effort when contestants are heterogeneous? The question can be examined in terms of either ex post *outcomes* or ex ante *contest rules*. The former concerns how contestants' winning odds are ranked in the optimum with respect to their innate strength, while the latter explores whether weaker contestants are necessarily favored vis-à-vis their stronger opponents. In Section 3.4, we generalize the conventional wisdom that with two players, the optimal contest handicaps the stronger and fully levels the playing field, i.e., $p_1^* = p_2^* = \frac{1}{2}$, regardless of the contest technology $h(\cdot)$. In an n-player lottery contest, Franke et al. (2013) show in a numerical example that the optimal contest is biased in favor of ex ante weaker players—i.e., $\alpha_i^* < \alpha_j^*$ for $v_i > v_j$, and $x_i^*, x_j^* > 0$, although the playing field is not fully balanced—i.e., $p_i^* > p_j^*$ for $v_i > v_j$, and $x_i^*, x_j^* > 0$. The conventional wisdom of leveling the playing field, however, has yet to be inspected in broader settings. Our exercise in this section fills the void.

4.1 Ranking of Winning Probabilities in the Optimum

In this part, we explore how the ranking of contestants' equilibrium winning odds in the effort-maximizing contest are related to their prize valuations. Recall the function $g(\cdot)$, which is defined as the inverse of $\log\big(c'(x)\cdot h(x)/h'(x)\big)$. We first obtain the following result.

Theorem 8 ***(Winning Probabilities in Effort-maximizing Contests)*** *Suppose that Assumption 1 is satisfied and the designer aims to maximize total effort. Consider a contest with $n \geq 3$. For two arbitrary active contestants $i, j \in \mathcal{N}$ with $v_i > v_j$, $p_i^* > p_j^*$ if $g(\cdot)$ is a strictly convex function.*

Theorem 8 predicts a monotone relationship between contestants' winning probabilities in the optimum and their prize valuations: For active contestants, a larger prize valuation ensures strictly higher equilibrium winning odds in the optimum when the function $g(\cdot)$ is convex.[22] Theorem 8 thus implies that a perfectly leveled playing field is a knife-edge case, as an artifact of bilateral competitions.

The formal proof is laid out in Appendix B, but the result, again, is straightforward in light of the fundamental correspondence:

$$x_i = g\left(\log\left(p_i(1-p_i)\right) + \log(v_i)\right), \forall\, i \in \mathcal{N}.$$

Obviously, x_i is supermodular in (p_i, v_i) when $g(\cdot)$ is strictly convex in its arguments: $\partial^2 x_i/\partial p_i \partial v_i$ must be strictly positive because by Theorem 6, $p_i^* < 1/2$ in the optimum. To interpret this more intuitively, $g(\cdot)$ depicts how a contestant's effort choice takes into account the value of prize and the prospect for his win: One steps up his effort when he expects a more rewarding prize (i.e., increasing v_i) or when he is more confident in his win (i.e., increasing p_i) for $p_i < 1/2$. The supermodularity implies that a brighter prospect for a win is more likely to incentivize a contestant when he also benefits more from the prize. The total effort can be maximized only when the assignment of $\boldsymbol{p}$ with respect to $\boldsymbol{v}$ is assortative, i.e., assigning larger equilibrium winning probability to a contestant of larger prize valuation.

By the same logic, the assignment pattern is set to be reversed when the function turns concave. It should be noted that $g(\cdot)$, in general, cannot be globally concave. To see that, recall that the function is the inverse of $\log\left(c'(x)\cdot h(x)/h'(x)\right)$. For a contest technology $h(\cdot)$ that satisfies Assumption 1 and a cost function $c(x_i)$ with a finite $c'(x_i)\big|_{x_i=0}$, $\log\left(c'(x)\cdot h(x)/h'(x)\right)$ approaches negative infinity in the neighborhood of zero, which precludes the possibility of global concavity of $g(\cdot)$. As a result, an exhaustive and complete comparative static of probability ranking is infeasible because the property of $g(\cdot)$ remains elusive in general.

In what follows, we construct a parameterized setting to illustrate the impact of $g(\cdot)$ on the monotonicity of the probability series in the optimum. To proceed, we assume a linear effort cost function $c(x) = x$, and parametrize the contest technology $h(\cdot)$ by a variable $\sigma \in (0, 1]$ as follows:

$$h_\sigma(x) := \exp\left(\int_1^x \frac{1}{\zeta_\sigma^{-1}(t)}dt\right),$$

where $\zeta_\sigma^{-1}(t)$ is the inverse function of $\zeta_\sigma(\cdot)$ given by

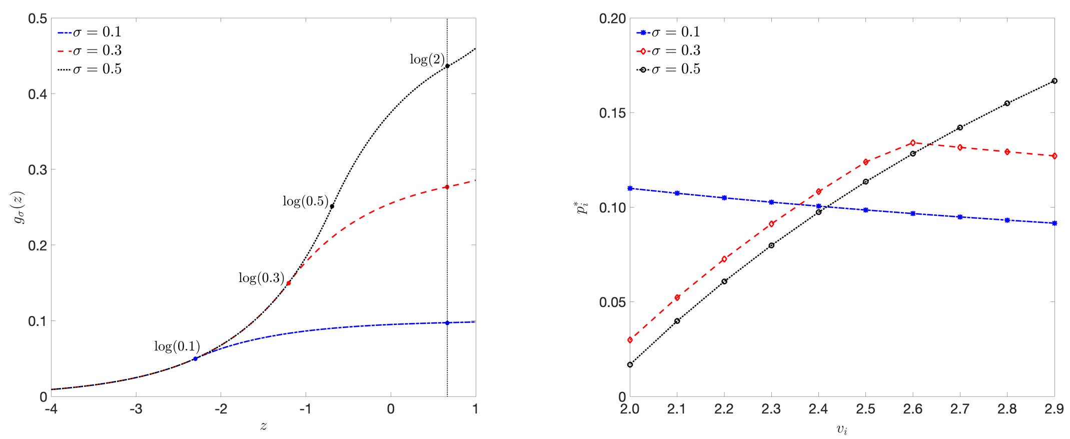
$$\zeta_\sigma(z) := \begin{cases} \frac{1}{2}z & \text{if } 0 < z < \sigma, \\ \sigma - \frac{\sigma^2}{2z} & \text{if } \sigma \leq z \leq 2, \\ \frac{\sigma^2}{8}z + (\sigma - \frac{1}{2}\sigma^2) & \text{if } z > 2. \end{cases}$$

[22]A convex $g(\cdot)$ is not uncommon. For instance, in a Tullock contest with $h(x_i) = (x_i)^r$ and a linear effort cost function, we can obtain that $g(z) = r\exp(z)$, which is evidently strictly convex.

It can be verified that the above contest technology $h_\sigma(x)$ satisfies Assumption 1. Carrying out the algebra, we can derive the expression of $g(\cdot)$, which we again index by σ, as

$$g_\sigma(z) = \zeta_\sigma(e^z) = \begin{cases} \frac{1}{2}e^z & \text{if } z < \log \sigma, \\ \sigma - \frac{\sigma^2}{2}e^{-z} & \text{if } \log \sigma \le z \le \log 2, \\ \frac{\sigma^2}{8}e^z + \left(\sigma - \frac{1}{2}\sigma^2\right) & \text{if } z > \log 2. \end{cases}$$

It is straightforward to see that $g_\sigma(z)$ is strictly convex in z for $z < \log \sigma$ and $z > \log 2$, and is strictly concave in z for $\log \sigma \le z \le \log 2$.



(a) $g_\sigma(z)$ under different levels of σ (b) $(p_1^*, \dots, p_{10}^*)$ under different levels of σ

Figure 1: $g_\sigma(z)$ and $(p_1^*, \dots, p_{10}^*)$ under Different Levels of σ.

Suppose that $n = 10$ and $(v_1, v_2, \dots, v_{10}) = (2.9, 2.8, \dots, 2.0)$. With a linear effort cost function $c(x) = x$ and the constructed contest technology $h_\sigma(\cdot)$, contestant i's first-order condition can now be rewritten as

$$p_i(1-p_i)v_i = \frac{h_\sigma(x_i)}{h'_\sigma(x_i)} = \zeta_\sigma^{-1}(x_i) \Rightarrow x_i = \zeta_\sigma\left(p_i(1-p_i)v_i\right).$$

Note that $p_i(1-p_i)v_i < 3/4 < 1$ in the example because $v_i < 3$ for all $i \in \mathcal{N} \equiv \{1, \dots, 10\}$. This, in turn, indicates that the region $[\log 1, \infty)$ in the support of $g_\sigma(\cdot)$ is irrelevant. The variable σ can therefore measure the concavity/convexity of the $g_\sigma(\cdot)$ function in the relevant support $(-\infty, \log 1)$, as Figure 1(a) depicts. In particular, $g_\sigma(\cdot)$ becomes globally concave in the relevant support as $\sigma \searrow 0$. Similarly, $g_\sigma(\cdot)$ is globally convex in the relevant support as $\sigma \nearrow 1$.

The profile of the optimal equilibrium probabilities $(p_1^*, \dots, p_{10}^*)$ for different values of σ are reported as follows:

σ	p_1^*	p_2^*	p_3^*	p_4^*	p_5^*	p_6^*	p_7^*	p_8^*	p_9^*	p_{10}^*
0.1	0.0915	0.0931	0.0948	0.0966	0.0985	0.1005	0.1026	0.1049	0.1073	0.1099
0.3	0.1271	0.1293	0.1316	0.1340	0.1239	0.1082	0.0912	0.0726	0.0522	0.0299
0.5	0.1668	0.1549	0.1421	0.1283	0.1134	0.0973	0.0798	0.0607	0.0398	0.0168

In the case of $\sigma = 0.5$, $p_i^* > p_j^*$ whenever $v_i > v_j$, as predicted by Theorem 8. In contrast, with $\sigma = 0.1$, $g_\sigma(\cdot)$ is globally concave in the relevant support and the ranking is entirely reversed. This implies that the optimal contest must severely handicap stronger contestants, such that they are less likely to win. The logic that underpins Theorem 8 can be flipped to interpret this observation. With a concave $g(\cdot)$, an increase in v_i reduces the marginal impact of p_i on x_i. A contestant can less effectively be motivated by an improvement in the prospect of a win when he has a higher valuation for the prize. As a result, a lower winning probability must be assigned to a contestant with a higher prize valuation, which prevents the marginal return of assigned winning probability from diminishing further. In the case of $\sigma = 0.3$, which is set in the middle of the two extremes, the ranking is nonmonotone. As Figure 1(b) illustrates, the equilibrium winning probability p_i^* strictly increases with i first and then decreases, with player 4 winning the contest with the highest probability.

4.2 Ranking of Multiplicative Biases in the Optimum

In this part, we examine the optimal contest rule—i.e., the multiplicative biases $\boldsymbol{\alpha}^*$—that maximizes total effort. Again, we construct a parameterized setting, assuming a Tullock contest with $n \geq 3$, $h(x_i) = (x_i)^r$, $r \in (0, 1]$, and a linear effort cost function $c(x_i) = x_i$. The parameter r measures the marginal return of a contestant's effort: With a larger r, a higher effort is more likely to be translated into larger winning odds. It is straightforward to verify that $g(z) = e^z$. By Theorem 8, contestants' winning odds p_i^* strictly decrease with i, in that an ex ante stronger contestant wins more often in the optimum.

The setting streamlines our analysis for two reasons. First, as stated above, the fundamental equilibrium correspondence under a Tullock contest setting can be simplified as

$$x_i = rp_i(1 - p_i)v_i, \forall\, i \in \mathcal{N},$$

which allows for a closed-form solution to the optimal bias rule $\boldsymbol{\alpha}^*$ as the optimization problem yields a simple quadratic programming. Second, the total effort of the contest can be rewritten as $\sum_{i=1}^n x_i = r\sum_{i=1}^n p_i(1-p_i)v_i$, which implies immediately that the optimal probability distribution $\boldsymbol{p}^*$, or the winning probability ranking in the optimum, is independent of the parameter r. This allows us to focus on the property of optimal contest rule and enables lucid comparative statics with respect to r. We first obtain the following theorem that fully characterizes the optimum.

Theorem 9 ***(Effort-maximizing Contests)*** *Assume without loss of generality that contestants are ordered such that $v_1 \geq v_2 \geq \ldots \geq v_n > 0$, $h(x_i) = (x_i)^r$, with $r \in (0,1]$, and $c(x_i) = x_i$. Suppose that the contest designer aims to maximize total effort. Then the equilibrium winning probabilities $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ are given by*

$$p_i^* = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{v_i} \times \frac{\kappa-2}{\sum_{j=1}^{\kappa} \frac{1}{v_j}}\right) & \text{if} \quad i \in \{1, \ldots, \kappa\}, \\ 0 & \text{if} \quad i \in \mathcal{N} \setminus \{1, \ldots, \kappa\}, \end{cases} \tag{12}$$

where κ is given by

$$\kappa := \max \left\{ m = 2, \ldots, n \,\middle|\, \frac{m-2}{\sum_{j=1}^{m} \frac{1}{v_j}} < v_m \right\}.$$

Moreover, the corresponding weights, denoted by $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$, that induce $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ are given by

$$\alpha_i^* = \begin{cases} \frac{\left(p_i^*\right)^{1-r}}{\left[\left(1-p_i^*\right)v_i\right]^r} & \text{if} \quad p_i^* > 0, \\ 0 & \text{if} \quad p_i^* = 0. \end{cases}$$

Theorem 9 allows us to rank $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ with respect to the parameter r.

Theorem 10 ***(Comparative Statics of the Optimal Biases with Respect to r)*** *Assume without loss of generality that contestants are ordered such that $v_1 \geq v_2 \geq \ldots \geq v_n > 0$, $h(x_i) = (x_i)^r$, with $r \in (0,1]$, and $c(x_i) = x_i$. Suppose that the contest designer aims to maximize total effort. Then the following statements hold:*

i. Suppose that contestants i and j remain active in the effort-maximizing contest (i.e., $i, j \leq \kappa$). If $v_i > v_j$, then there exists a cutoff $\overline{r}_{ij} \in (0,1)$ such that $\alpha_i^* \gtreqless \alpha_j^*$ if $r \lesseqgtr \overline{r}_{ij}$.

ii. Define an upper bound $\overline{r}_{\max} := \max_{\{i<j\leq\kappa\}}\{\overline{r}_{ij}\}$ and a lower bound $\overline{r}_{\min} := \min_{\{i<j\leq\kappa\}}\{\overline{r}_{ij}\}$. α_m^* is decreasing in $m \in \{1, \ldots, \kappa\}$ when $r \leq \overline{r}_{\min}$, and is increasing when $r \geq \overline{r}_{\max}$. For $r \in (\overline{r}_{\min}, \overline{r}_{\max})$, the optimal biases $\boldsymbol{\alpha}^*$ are nonmonotone.

Theorem 10 indicates that the usual leveling-the-playing-field principle does not hold in general. It first states that for a given pair of active contestants, the optimal bias rule can favor either of them depending on the size of r. More generally, Theorem 10(ii) identifies two cutoffs. When the contest sufficiently rewards more effort, i.e., $r \geq \overline{r}_{\max}$, a larger weight is assigned to a weaker active player, i.e., one with a lower prize valuation, in which case the conventional wisdom remains. In contrast, when r falls below a lower bound $\overline{r}_{\min}$, the prediction is entirely reversed, and the designer further upsets the balance of the contest in the optimum, i.e., α_m^* is decreasing in m: The optimal contest favors ex ante stronger contestants.[23,24] When r falls in the intermediate range $(\overline{r}_{\min}, \overline{r}_{\max})$, the ranking of $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ is no longer monotone.

We construct a numerical example to illustrate the comparative statics. Again, suppose that $n = 10$ and $(v_1, v_2, \ldots, v_{10}) = (2.9, 2.8, \ldots, 2.0)$. To ease comparison with respect to r, we normalize the sum of optimal weights established by Theorem 9 to one and define $\alpha_i' \equiv \alpha_i^*/(\sum_{j=1}^n \alpha_j^*)$ for all $i \in \mathcal{N} \equiv \{1, \ldots, 10\}$.[25] The optimal bias rule for a given r can then be identified as follows:

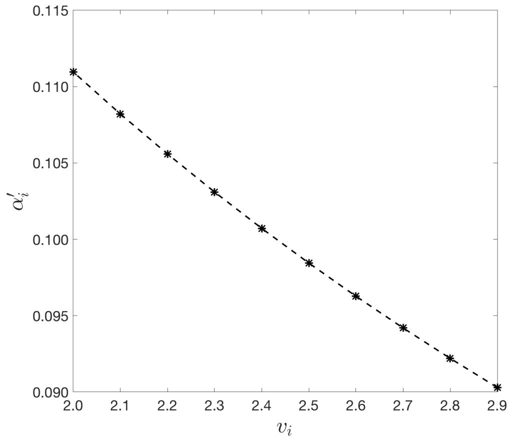
r	α_1'	α_2'	α_3'	α_4'	α_5'	α_6'	α_7'	α_8'	α_9'	α_{10}'
1.0	0.0903	0.0922	0.0942	0.0963	0.0984	0.1007	0.1031	0.1056	0.1082	0.1110
0.9	0.0979	0.0990	0.1001	0.1010	0.1018	0.1023	0.1025	0.1019	0.0998	0.0937
0.4	0.1364	0.1316	0.1260	0.1196	0.1121	0.1032	0.0925	0.0792	0.0621	0.0374

We illustrate the three cases in Figure 2. Monotone rankings of $(\alpha_1', \ldots, \alpha_{10}')$ arise in the case of both a large r ($r = 1$) and a small r ($r = 0.4$): The former exemplifies the conventional wisdom of leveling the playing field, while the latter entirely defies that. In the case of intermediate r ($r = 0.9$), contestant 7, with a prize valuation 2.3, receives the most favoritism from the designer [see Figure 2(b)]: The optimal contest levels the playing field for contestants 1-7, but discounts the output of the weakest three. The second panel of Figure 2 depicts the case of nonmonotone ranking. The curve that traces α_m' with respect to contestants' prize valuation v_m is inverted U-shaped.

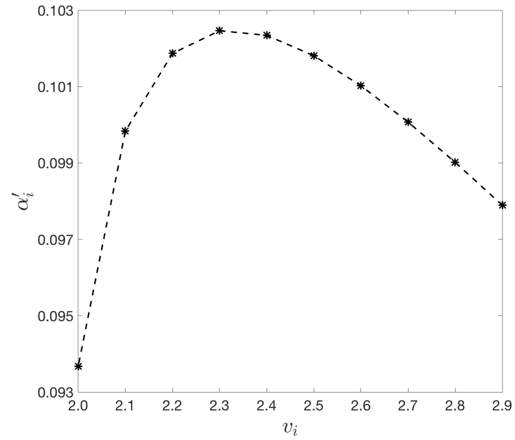
[23] Reverse handicapping in favor of ex ante stronger contenders is not uncommon in reality. Consider, for instance, the widely practiced industry policy that gives unfair advantage to large organizations to promote "national champions" for domestic dominance and international preeminence; e.g., the dirigiste policy in France from 1945 to 1947 and Korea's industrialization programs. Alternatively, the financial fair-play regulation (FFP) in European football (soccer) has been broadly criticized for the anticompetition role it played to perpetuate the dominance of "big clubs": The rule requires that European football clubs balance their books and not spend more than the income they generate, which solidifies an incumbent "big" club's advantage in attracting talent, given the superior revenue it receives based on its past track record.

[24] Soccer is broadly viewed as the least predictable major sporting discipline. Ben-Naim, Vazquez and Redner (2007) and Anderson and Sally (2013) provide extensive empirical evidence that soccer matches produced "upsets"—i.e., pregame underdogs overcoming favorites—more frequently than other sports, which alludes to a relatively more significant role played by luck in soccer matches vis-à-vis skill or effort. Our result can thus arguably shed light on the European FFP regulation that advantages big clubs (see Footnote 23). This stands in contrast to various measures in the NBA—e.g., the draft lottery and salary cap—that maintain a level playing field. Anderson and Sally, among others, show that the results of basketball matches are the most predictable based on teams' quality (see https://knowledge.wharton.upenn.edu/article/sports-by-the-numbers-predicting-winners-and-losers/).

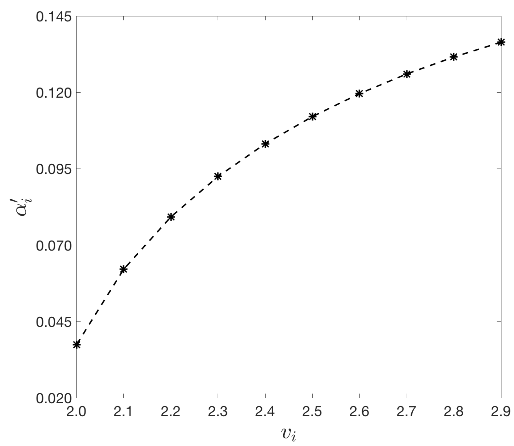
[25] The variable α_i' can be interpreted as contestant i's winning probabilities if all contestants exert the same amount of effort.



(a) $r = 1$



(b) $r = 0.9$



(c) $r = 0.4$

Figure 2: Optimal Effort-Maximizing Weighting Rule under Different Levels of r.

The optimal bias rule subtly depends on the various environmental factors of the contest, e.g., the parameter r. However, the comparative statics can again be interpreted in light of the fundamental correspondence and our optimization approach. As stated above, $\boldsymbol{p}^*$, the winning probability distribution in the optimum, must remain constant regardless of r. Imagine that r decreases. A higher effort—contributed by a stronger contestant—can be less effectively converted into higher winning odds, which narrows the spread in $\boldsymbol{p}^*$ and, in turn, depletes contestants' effort incentives. To counteract this effect and restore the required distribution $\boldsymbol{p}^*$, an ex ante stronger contestant must be handicapped less severely because a larger α_i imposed on an ex ante stronger contestant, ceteris paribus, tends to enlarge the spread in the distribution of winning probabilities for any given effort profile.

More intuitively, recall the usual rationale for leveling the playing field: Favoritism motivates the underdog, which in turn prevents the favorite from slacking off. This rationale can be cast into doubt when r decreases. A smaller r diminishes all contestants' incentives. On the one hand, a weaker contestant would respond less sensitively in his effort choice to the extra favoritism he receives. On the other hand, a stronger contestant tends to be less privileged to slack off regardless: A smaller r erodes his advantage because his higher effort is less effective for securing larger winning odds. As a result, a contest rule in favor of the weak loses its appeal, as both the positive incentive effect for underdogs and the disciplinary effect on the favorite tend to fade away. Handicapping strong contestants may backfire, as it excessively suppresses their winning odds and mutes their incentives. The optimum could turn out to favor favorites more to counteract these effects and preserve their momentum.

5 Concluding Remarks

In this paper, we develop a novel optimization approach to study the design of biased contests. A designer is allowed to impose identity-dependent preferential treatments on heterogeneous contestants. A closed-form solution to the equilibrium of an n-player asymmetric contest is, in general, unavailable, which nullifies the usual implicit programming approach. Our approach allows us to bypass the analytical difficulty. Based on a fundamental correspondence derived from the equilibrium condition, we reformulate the optimization problem and are able to characterize the general properties of the optimal contest in a substantially generalized setting with flexible contest technology, noncanonical objective functions, and an arbitrary number of players. The analysis enabled by the approach generates useful theoretical implications that contrast starkly with those obtained in the restricted settings considered in previous studies. In particular, we demonstrate that the conventional wisdom of leveling the playing field may not hold in general. The optimum could even require that the contest rules favor ex ante stronger contestants vis-à-vis their weaker opponents. In addition to its technical contribution, the approach, based on the aforementioned fundamental correspondence, sheds light on the nature of incentive provision in contests.

Our paper assumes that contestants are endowed with the same contest technology $h(\cdot)$ and effort cost function $c(\cdot)$. Notably, many of our results do not depend on this modeling nuance. More specifically, Theorems 2-3 and 5-7 are qualitatively unchanged when the restrictions are relaxed. Encapsulating contestants' heterogeneity into the difference in their prize valuations, however, provides a convenient measure or definition of contestants' strength and allows for lucid comparative statics, which gives rise to Theorems 4, 8, and 10.

Our approach substantially eases the analysis of optimal contest design and can be applied to a broad array of scenarios. Assuming a Tullock contest technology, the approach yields a closed-form solution to the optimal contest rule for a large class of objective functions, which enables future research on specific issues. For instance, it can be applied to reexamine the classical issue of comparing all-pay auctions and lottery contests when alternative design objectives other than total effort maximization are in place. Our paper considers a static contest, but the approach can also be applied in dynamic settings. For instance, Fu and Wu (2019b) consider a two-stage contest in which the designer assigns individualized weights to contestants' second-stage effort entries based on their first-stage ranking. Finally, this paper focuses on contests with complete information; it would be interesting to extend the analysis to an environment with incomplete information, which should be attempted in future research.

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Appendix A Microfoundation

We interpret the microeconomic substance of the generalized lottery contest model from two perspectives.

Noisy Ranking Clark and Riis (1996) and Jia (2008) show that a generalized lottery contest is underpinned by a unique noisy ranking system. Imagine that contestants are evaluated through a set of noisy signals of their performance ℓ_is. Following the discrete choice framework of McFadden (1973, 1974),[26] the noisy signal ℓ_i is assumed to be described by

$$\log \ell_i = \log f_i(x_i) + \varepsilon_i, \forall\, i \in \mathcal{N},$$

where the deterministic and strictly increasing production function $f_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ measures the output of contestant i's effort x_i,[27] and the additive noise term ε_i reflects the randomness in the production process or the imperfection of the measurement and evaluation process. Idiosyncratic noises $\boldsymbol{\varepsilon} := \{\varepsilon_i, i \in \mathcal{N}\}$ are independently and identically distributed, being drawn from a type I extreme-value (maximum) distribution, with a cumulative distribution function

$$G(\varepsilon_i) = e^{-e^{-\varepsilon_i}}, \varepsilon_i \in (-\infty, +\infty), \forall\, i \in \mathcal{N}.$$

A contestant i prevails if he outperforms all others: This noisy-ranking tournament boils down to a generalized lottery contest, because

$$\Pr\left(\ell_i > \max_{j \neq i} \ell_j\right) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}.$$

Isomorphism to R&D Contests Baye and Hoppe (2003) demonstrate the isomorphism between a generalized lottery contest, the research tournament model proposed by Fullerton and McAfee (1999), and the patent race model suggested by Loury (1979) and Dasgupta and Stiglitz (1980). This provides a more intuitive microeconomic underpinning for the model.

To illustrate the equivalence, we focus on the research tournament model of Fullerton and McAfee (1999). A sponsor—who is interested in an innovative technology—invites $n \geq 2$ R&D firms to carry out the project. Firms develop the technology and submit their products to the designer. The entry of the highest quality wins and its developer is awarded a prize, such as a procurement contract. Each firm i's valuation of the prize is given by $v_i > 0$.

Each firm i decides on its own input $x_i \geq 0$ in developing the technology. The quality

[26] The framework of McFadden's discrete choice model is further introduced and studied in various respects by works collected in Manski and McFadden (1981).

[27] Define $\log f_i(x_i) = -\infty$ if $f_i(x_i) = 0$.

q_i of firm i's product is randomly drawn from a distribution with cumulative distribution function $\left[\Gamma(q_i)\right]^{f_i(x_i)}$. The function $\Gamma(\cdot)$ is a continuous cumulative distribution function on a support $\left[\underline{q}, \overline{q}\right]$, with $\overline{q} > \underline{q}$. By Fullerton and McAfee (1999) and Baye and Hoppe (2003), the term $f_i(x_i)$—which increases with x_i—can intuitively be interpreted as the number of research ideas generated in developing the product and indicates the firm's research capacity: Each research idea allows the firm to produce a prototype, with its quality being drawn from the distribution function $\Gamma(\cdot)$. A firm simply presents its best prototype to the sponsor as its entry, and the quality of its entry thus follows the distribution function $\left[\Gamma(q_i)\right]^{f_i(x_i)}$: The more ideas a firm generates, the more likely a higher q_i can be realized, and the more likely the firm can leapfrog its competitors. As pointed out by Baye and Hoppe (2003) and Fu and Lu (2012), a firm i wins the prize with a probability

$$\Pr\left(q_i > \max_{j \neq i} q_j\right) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}.$$

A similar equivalence can be established between a generalized lottery contest model and the "first past the post" patent race model of Loury (1979) and Dasgupta and Stiglitz (1980), in which a firm secures a rent if it makes a scientific discovery earlier than its competitors. Fu and Lu (2012) further reveal the underlying statistical linkage between these R&D contests and the generalized lottery contest model (1).

Appendix B Proofs

Proof of Theorem 1

Proof. Note that $x_i = 0$ is a strictly dominant strategy for contestant i if $f_i(\cdot)$ is a constant. Therefore, it suffices to prove the theorem for the case in which $f_i(\cdot)$ satisfies $f_i'(x_i) > 0$, $f_i''(x_i) \leq 0$ and $f_i(0) \geq 0$ for all $i \in \mathcal{N}$.

For notational convenience, define $y_i := f_i(x_i)$, $\delta_i := f_i(0)$, $\widetilde{f}_i(x_i) := f_i(x_i) - \delta_i$, and $\lambda_i(y_i) := c\big(\widetilde{f}_i^{-1}(y_i - \delta_i)\big)/v_i$. It follows immediately that $c(x_i) = \lambda_i(y_i) \cdot v_i$. Moreover, we have that $\lambda_i' > 0$ and $\lambda_i'' \geq 0$. The expected payoff of contestant $i \in \mathcal{N}$ choosing $y_i \geq \delta_i$ is equal to

$$\left[\frac{y_i}{\sum_{j=1}^n y_j} - \lambda_i(y_i)\right] \cdot v_i.$$

It remains to show that there exists a unique equilibrium $\boldsymbol{y}^* \equiv (y_1^*, \ldots, y_n^*)$ that satisfies $y_i^* \geq \delta_i$ for all $i \in \mathcal{N}$. Let $s := \sum_{j=1}^n y_j$ and $\underline{\delta} := \sum_{j=1}^n \delta_j$. It is clear that $s \geq \underline{\delta}$. The first-order condition of the above expected utility with respect to y_i yields the following:

$$\frac{s - y_i}{s^2} - \lambda_i'(y_i) \leq 0, \text{ with equality if } y_i > \delta_i.$$

Fixing s, let us define $y_i(s)$ as the following:

$$y_i(s) := \begin{cases} \delta_i & \text{if } s^2\lambda_i'(\delta_i) - s + \delta_i \geq 0, \\ \text{The unique solution to } s - y_i = s^2\lambda_i'(y_i) & \text{otherwise.} \end{cases} \tag{A1}$$

It is straightforward to verify that $y_i(s)$ is well defined and continuous in $s \in [\delta_i, \infty]$. Moreover, we must have that $y_i(s) \in (\delta_i, s)$ if $s^2\lambda_i'(\delta_i) - s + \delta_i < 0$.

Suppose that there exists an interval of s such that $y_i(s) > \delta_i$. It follows immediately from the implicit function theorem that

$$y_i'(s) = \frac{1 - 2s\lambda_i'(y_i)}{1 + s^2\lambda_i''(y_i)} = \frac{2y_i(s) - s}{\left[1 + s^2\lambda_i''(y_i)\right] s}, \tag{A2}$$

where the second equality follows from $s - y_i = s^2\lambda_i'(y_i)$. Therefore, $y_i(s)$ is strictly decreasing in this interval if $2y_i < s$ and strictly increasing otherwise. By Equation (A1), the latter case occurs if and only if

$$s - \frac{1}{2}s > s^2\lambda_i'\left(\frac{s}{2}\right) \Leftrightarrow 2s\lambda_i'\left(\frac{s}{2}\right) < 1.$$

Note that $2s\lambda_i'\left(\frac{s}{2}\right)$ is strictly increasing in s, which implies that there exists at most one solution to $2s\lambda_i'\left(\frac{s}{2}\right) = 1$. Denote the solution by $\hat{s}_i$ whenever it exists.

Next, we denote the two different real number solutions of $s^2\lambda_i'(\delta_i) - s + \delta_i = 0$ by $s_i^{\dagger}$ and $s_i^{\dagger\dagger}$ respectively, with $s_i^{\dagger} < s_i^{\dagger\dagger}$, whenever they exist. The above analysis, together with the fact that the expression $s^2\lambda_i'(\delta_i) - s + \delta_i$ in Equation (A1) is quadratic in s, implies that the function $y_i(s)$ must fall into one of the following four cases:

Case I: There exist no different real number solutions of $s^2\lambda_i'(\delta_i) - s + \delta_i = 0$ for $s \in [\underline{\delta}, \infty]$. Then we must have that $s^2\lambda_i'(\delta_i) - s + \delta_i \geq 0$ for all $s \geq \underline{\delta}$, which in turn implies that $y_i(s) = \delta_i$ for all $s \geq \underline{\delta}$ by Equation (A1). To slightly abuse the notation, we let $s_i^{\dagger\dagger} := \underline{\delta}$ for this case.

Case II: $s_i^{\dagger} \leq \underline{\delta} \leq s_i^{\dagger\dagger}$ and $y_i(\underline{\delta}) \leq \frac{1}{2}\underline{\delta}$. Then $y_i(s)$ is strictly decreasing in s for $s \in [\underline{\delta}, s_i^{\dagger\dagger}]$, and $y_i(s) = \delta_i$ for $s \in [s_i^{\dagger\dagger}, \infty]$.

Case III: $s_i^{\dagger} \leq \underline{\delta} \leq s_i^{\dagger\dagger}$ and $y_i(\underline{\delta}) > \frac{1}{2}\underline{\delta}$. It can be verified that $\underline{\delta} < \hat{s}_i < s_i^{\dagger\dagger}$. Therefore, $y_i(s)$ is strictly increasing in s for $s \in [\underline{\delta}, \hat{s}_i]$; is strictly decreasing in s for $s \in [\hat{s}_i, s_i^{\dagger\dagger}]$; and $y_i(s) = \delta_i$ for $s \in [s_i^{\dagger\dagger}, \infty]$.

Case IV: $\underline{\delta} < s_i^{\dagger} < s_i^{\dagger\dagger}$. It can be verified that $s_i^{\dagger} < \hat{s}_i < s_i^{\dagger\dagger}$. Moreover, $y_i(s)$ is strictly increasing in s for $s \in [s_i^{\dagger}, \hat{s}_i]$; is strictly decreasing in s for $s \in [\hat{s}_i, s_i^{\dagger\dagger}]$; and $y_i(s) = \delta_i$ for $s \in [\underline{\delta}, s_i^{\dagger}] \cup [s_i^{\dagger\dagger}, \infty]$.

The four cases are depicted in Figure 3 graphically. For Case I and Case II, we define $s_i := \underline{\delta}$; for Case III and Case IV, we define $s_i := \hat{s}_i \geq \underline{\delta}$. It is straightforward to verify that $y_i(s) > \frac{1}{2}s$ holds if $s < s_i$ for all four cases. Without loss of generality, we order the contestants such that

$$s_1 \geq s_2 \geq \ldots \geq s_n \geq \underline{\delta}.$$

Define $Y(s) := \sum_{i=1}^{n} y_i(s) - s$. It remains to show that $Y(s) = 0$ has a unique positive solution. First, note that no solution exists for $s < s_2$, because

$$Y(s) := \sum_{i=1}^{n} y_i(s) - s \geq y_1(s) + y_2(s) - s > \frac{1}{2}s + \frac{1}{2}s - s = 0, \text{ for } s < s_2.$$

Next, we claim that $Y(s)$ is strictly decreasing in s for $s \geq s_2$. Clearly, $Y(s)$ is strictly decreasing in s for $s \geq s_1$. Moreover, for $s \in [s_2, s_1]$, $Y(s)$ can be rewritten as

$$Y(s) = \underbrace{\sum_{i=2}^{n} y_i(s)}_{\text{first term}} + \underbrace{[y_1(s) - s]}_{\text{second term}}.$$

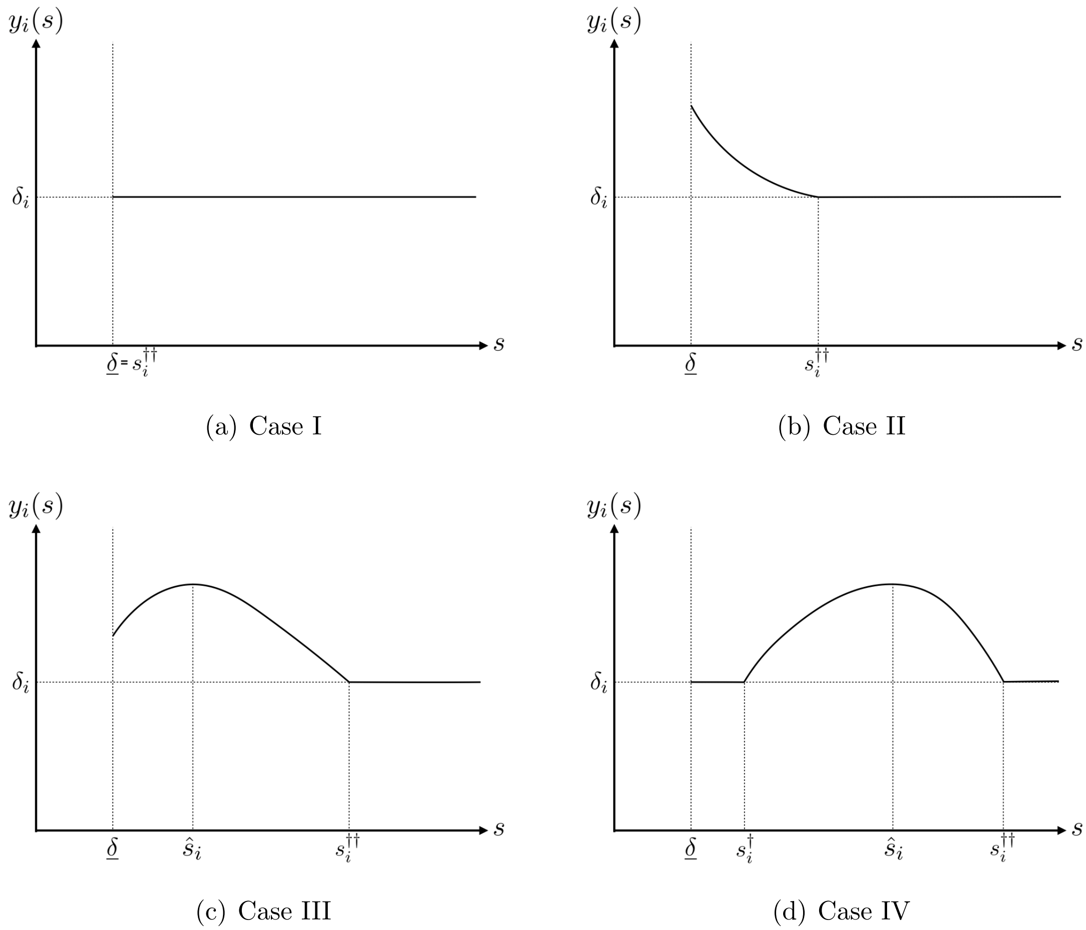

Figure 3: $y_i(s)$.

Because $s \geq s_2 \geq \ldots \geq s_n$, the first term is weakly decreasing in s. Taking the derivative of the second term with respect to s yields

$$y_1'(s) - 1 = \frac{2y_1(s) - s}{\left[1 + s^2 \lambda_1''\left(y_1(s)\right)\right] s} - 1 \leq \frac{2y_1(s) - s}{s} - 1 = \frac{2}{s}\left[y_1(s) - s\right] < 0,$$

where the first equality follows from Equation (A2); the first inequality follows from $\lambda_i'' \geq 0$ and $y_1(s) \geq \frac{s}{2}$, and the second inequality follows from $y_i(s) < s$ [see Eq uation (A1)]. Therefore, the second term is strictly decreasing in s, which in turn implies that $Y(s)$ is strictly decreasing for $s \in [s_2, \infty]$.

It is straightforward to see that for all four cases, we have that $y_i(s) = \delta_i$ for $s \geq s_i^{\dagger\dagger}$. Let

$s^{\dagger\dagger} := s_2 + \sum_{i=1}^n s_i^{\dagger\dagger} + \sum_{i=1}^n \delta_i$. It is clear that $s^{\dagger\dagger} \geq s_2$. Moreover, we have that

$$Y(s^{\dagger\dagger}) = \sum_{i=1}^n y_i(s^{\dagger\dagger}) - s^{\dagger\dagger} = \sum_{i=1}^n \delta_i - \left(s_2 + \sum_{i=1}^n s_i^{\dagger\dagger} + \sum_{i=1}^n \delta_i \right) = -s_2 - \sum_{i=1}^n s_i^{\dagger\dagger} \leq 0.$$

Therefore, there exists a unique positive solution to $Y(s) = 0$ for $s \in [s_2, s^{\dagger\dagger}]$. This completes the proof. ■

Proof of Theorem 2

Proof. The analysis for the case $x_t^* > 0$ is provided in the main text, and it suffices to prove the theorem for the case $x_t^* = 0$. Because $\beta_t^* > 0$, we must have $p_t^* > 0$. If $p_t^* = 1$, then we must have $\boldsymbol{x}^* = \mathbf{0}$. Clearly, the equilibrium outcome (i.e., $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$) can be replicated by the following contest rule with zero headstarts:

$$(\alpha_i, \beta_i) := \begin{cases} (1, 0) & \text{for} \quad i = t, \\ (0, 0) & \text{for} \quad i \neq t. \end{cases}$$

Therefore, it remains to focus on the case in which $p_t^* \in (0, 1)$. Denote by $x^{\dagger\dagger}$ the unique solution to the following equation:

$$p_t^*(1 - p_t^*) v_t = c'(x^{\dagger\dagger}) \cdot \frac{h(x^{\dagger\dagger})}{h'(x^{\dagger\dagger})}.$$

Note that the left-hand side of the above equation is strictly positive. Therefore, $x^{\dagger\dagger} > 0 = x_t^*$. Consider the following contest rule with weights $\widehat{\boldsymbol{\alpha}} \equiv (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_n)$ and headstarts $\widehat{\boldsymbol{\beta}} \equiv (\widehat{\beta}_1, \ldots, \widehat{\beta}_n)$ such that

$$\left(\widehat{\alpha}_i, \widehat{\beta}_i\right) := \begin{cases} \left(\frac{\alpha_t^* h(x_t^*) + \beta_t^*}{h(x^{\dagger\dagger})}, 0 \right) & \text{for} \quad i = t, \\ (\alpha_i^*, \beta_i^*) & \text{for} \quad i \neq t. \end{cases}$$

Denote the equilibrium effort profile and winning probabilities under the alternative contest rule $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ by $\widehat{\boldsymbol{x}}^* \equiv (\widehat{x}_1^*, \ldots, \widehat{x}_n^*)$ and $\widehat{\boldsymbol{p}}^* \equiv (\widehat{p}_1^*, \ldots, \widehat{p}_n^*)$, respectively. It can be verified that

$$\widehat{x}_i^* = \begin{cases} x^{\dagger\dagger} & \text{for} \quad i = t, \\ x_i^* & \text{for} \quad i \neq t. \end{cases}$$

Moreover, we have that $\widehat{p}_i^* = p_i^*$ for all $i \in \mathcal{N}$ because $\widehat{\alpha}_t \cdot h(x^{\dagger\dagger}) + \widehat{\beta}_t = \alpha_t^* \cdot h(x_t^*) + \beta_t^*$ by construction. Therefore, the contest designer's payoff under $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ is weakly higher than that under $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ by Assumption 2. This completes the proof. ■

Proof of Theorem 3

Proof. Part (i) of the lemma is trivial, and it remains to show part (ii). It is clear that $x_i = 0$ is a strictly dominant strategy if $\alpha_i = 0$. For $(p_i, p_j) > (0, 0)$, we must have $(x_i, x_j) > (0, 0)$. Therefore, the following first-order conditions must be satisfied by Equation (9):

$$\begin{aligned} x_i &= g\Big(\log(p_i\,(1-p_i)) + \log(v_i)\Big), \\ x_j &= g\Big(\log(p_j\,(1-p_j)) + \log(v_j)\Big). \end{aligned}$$

Note that Equation (1) implies that

$$\frac{p_i}{p_j} = \frac{\frac{\alpha_i \cdot h(x_i)}{\sum_{k=1}^n \alpha_k \cdot h(x_k)}}{\frac{\alpha_j \cdot h(x_j)}{\sum_{k=1}^n \alpha_k \cdot h(x_k)}} = \frac{\alpha_i \cdot h(x_i)}{\alpha_j \cdot h(x_j)}.$$

Combining the above conditions, we can obtain that

$$\frac{\alpha_i}{\alpha_j} = \frac{p_i/h(x_i)}{p_j/h(x_j)} = \frac{\frac{p_i}{h\left(g\left(\log(p_i(1-p_i))+\log(v_i)\right)\right)}}{\frac{p_j}{h\left(g\left(\log(p_j(1-p_j))+\log(v_j)\right)\right)}}.$$

The last equation clearly holds for the set of weights specified in Equation (11). This completes the proof. ■

Proof of Theorem 4

Proof. With slight abuse of notation, let us define $x(p_k, v_k) := g\Big(\log(p_k\,(1-p_k)) + \log(v_k)\Big)$. Then the equilibrium effort x_k in Equation (9) can be written as $x(p_k, v_k)$ for all $k \in \mathcal{N}$. Define $\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v}) := \big(x(p_1, v_1), \ldots, x(p_n, v_n)\big)$. It follows immediately that $\tau\left(\boldsymbol{x}(\boldsymbol{p}, \boldsymbol{v})\right) = \boldsymbol{x}\left(\tau(\boldsymbol{p}), \tau(\boldsymbol{v})\right)$. Moreover, Equation (9) implies that $x(0, v) = 0$ for all $v > 0$.

Suppose, to the contrary, that there exists some contestant $j \in \mathcal{N}$ with $v_j < v_i$ such that $p_i^* = 0 < p_j^*$. Then we can obtain

$$\begin{aligned} \Lambda\left(\boldsymbol{x}(\boldsymbol{p}^*, \boldsymbol{v}), \boldsymbol{p}^*, \boldsymbol{v}\right) &\leq \Lambda\left(\boldsymbol{x}(\boldsymbol{p}^*, \boldsymbol{v}), \boldsymbol{p}^*, \tau_{ij}(\boldsymbol{v})\right) \\ &= \Lambda\Big(\tau_{ij}\left(\boldsymbol{x}(\boldsymbol{p}^*, \boldsymbol{v})\right), \tau_{ij}(\boldsymbol{p}^*), \boldsymbol{v}\Big) \\ &= \Lambda\Big(\boldsymbol{x}\left(\tau_{ij}(\boldsymbol{p}^*), \tau_{ij}(\boldsymbol{v})\right), \tau_{ij}(\boldsymbol{p}^*), \boldsymbol{v}\Big) \\ &< \Lambda\Big(\boldsymbol{x}\left(\tau_{ij}(\boldsymbol{p}^*), \boldsymbol{v}\right), \tau_{ij}(\boldsymbol{p}^*), \boldsymbol{v}\Big). \end{aligned}$$

The first inequality follows from $x(p_i^*, v_i) = 0$ and part (ii) of Assumption 3; the first equality follows from part (i) of Assumption 3 and the fact that $\tau_{ij}\left(\tau_{ij}(\boldsymbol{v})\right) = \boldsymbol{v}$; the second equality follows from $\tau_{ij}\left(\boldsymbol{x}(\boldsymbol{p}^*, \boldsymbol{v})\right) = \boldsymbol{x}\left(\tau_{ij}(\boldsymbol{p}^*), \tau_{ij}(\boldsymbol{v})\right)$; and the last strict inequality follows from $x(p_i^*, v_i) = x(p_i^*, v_j) = 0$, $x(p_j^*, v_j) < x(p_j^*, v_i)$, the postulated $p_j^* > 0$, and part (iii) of Assumption 3. Therefore, the contest designer's payoff under the optimal vector of winning probabilities $\boldsymbol{p}^*$ is strictly lower than that under $\tau_{ij}(\boldsymbol{p}^*)$, which is a contradiction. This completes the proof. ■

Proof of Theorem 5

Proof. See the main text. ■

Proof of Theorem 6

Proof. We first prove part (i) of the theorem. Suppose, to the contrary, that only two players remain active in the optimal contest. It is clear that $p_1^* = p_2^* = \frac{1}{2}$ in the optimum. Consider the following profile of equilibrium winning probabilities $\boldsymbol{p} = (\frac{1}{2}, \frac{1}{2} - \epsilon, \epsilon, 0, \ldots, 0)$. It can be verified that the total effort under $\boldsymbol{p}$ is equal to

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}) = g\left(\log(\frac{1}{4}) + \log(v_1)\right) + g\left(\log(\frac{1}{4} - \epsilon^2) + \log(v_2)\right) + g\left(\left(\log(\epsilon(1-\epsilon)\right) + \log(v_3)\right).$$

Simple algebra shows that $\partial\Lambda/\partial\epsilon > 0$ when ϵ is sufficiently small. Therefore, at least three players will remain active in the optimum.

Next, we prove part (ii). Suppose, to the contrary, that $p_i^* \geq \frac{1}{2}$ for some $i \in \mathcal{N}$. If $p_i^* > \frac{1}{2}$, then the contest designer can assign probability $1 - p_i^*$ to contestant i and probability $p_j^* + (2p_i^* - 1)$ to an arbitrary contestant $j \neq i$. Because at least three players remain active in the optimum, we must have $p_i^* + p_j^* < 1$. This in turn implies that $|p_j^* + (2p_i^* - 1) - \frac{1}{2}| < |p_j^* - \frac{1}{2}|$, and thus contestant j's effort strictly increases. Furthermore, it follows from Equation (9) that contestant i's effort remains the same. Therefore, the total effort strictly increases after the adjustment. If $p_i^* = \frac{1}{2}$, then there exists an active player $j \in \mathcal{N}$ such that $p_j \in (0, \frac{1}{2})$, because at least three players remain active in the optimum. In such a scenario, the designer can increase the total effort by reducing p_i^* by a sufficiently small amount and increasing p_j^* by the same amount. This completes the proof. ■

Proof of Theorem 7

Proof. It is useful to first prove the following intermediate result.

Lemma A1 *Consider a contest with three players who are indexed by i, j, and k. Suppose that the contest designer aims to maximize the expected winner's effort. Then setting $p_i = p_j = p_k = \frac{1}{3}$ is suboptimal.*

Proof. Without loss of generality, we assume that $v_i \geq v_j \geq v_k$. The difference between the expected winner's effort under $(p_i, p_j, p_k) = (\frac{1}{2}, \frac{1}{2}, 0)$ and that under $(p_i, p_j, p_k) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ can be derived as

$$
\begin{aligned}
&\left[\frac{1}{2}g\left(\log\left(\frac{1}{4}\right) + \log(v_i)\right) + \frac{1}{2}g\left(\log\left(\frac{1}{4}\right) + \log(v_j)\right)\right] \\
&\quad - \left[\frac{1}{3}g\left(\log\left(\frac{2}{9}\right) + \log(v_i)\right) + \frac{1}{3}g\left(\log\left(\frac{2}{9}\right) + \log(v_j)\right) + \frac{1}{3}g\left(\log\left(\frac{2}{9}\right) + \log(v_k)\right)\right] \\
&> \frac{1}{6}\left[g\left(\log\left(\frac{2}{9}\right) + \log(v_i)\right) - g\left(\log\left(\frac{2}{9}\right) + \log(v_j)\right)\right] \\
&\geq 0,
\end{aligned}
$$

where the strict inequality follows from $\frac{1}{4} > \frac{2}{9}$, $v_j \geq v_k$, and the monotonicity of $g(\cdot)$. Therefore, setting $p_i = p_j = p_k = \frac{1}{3}$ is suboptimal. This completes the proof. ■

Now we can prove the theorem. Suppose, to the contrary, that three or more players remain active in the optimal contest. Then there exist $i, j, k \in \mathcal{N}$ such that $p_i^{**} \geq p_j^{**} > 0$ and $p_i^{**} \geq p_k^{**} > 0$. Lemma A1 implies that $\min\{2p_j^{**} + p_k^{**}, p_j^{**} + 2p_k^{**}\} < 1$. Without loss of generality, we assume that $v_j \geq v_k$.

Suppose that the contest designer assigns probability $p_{jk}^{**} := p_j^{**} + p_k^{**}$ to player j and 0 to player k, and does not change the equilibrium winning probability of all other players. Then the difference between the expected winner's effort under the new profile of winning probabilities and that under $\boldsymbol{p}^{**} \equiv (p_1^{**}, \ldots, p_n^{**})$ can be derived as

$$
\begin{aligned}
&(p_j^{**} + p_k^{**}) g \left(\log \left(p_{jk}^{**}(1 - p_{jk}^{**}) \right) + \log(v_j) \right) \\
&- \left[p_j^{**} g \left(\log \left(p_j^{**}(1 - p_j^{**}) \right) + \log(v_j) \right) + p_k^{**} g \left(\log \left(p_k^{**}(1 - p_k^{**}) \right) + \log(v_k) \right) \right] \\
=& \, p_j^{**} \left[g \left(\log \left(p_{jk}^{**}(1 - p_{jk}^{**}) \right) + \log(v_j) \right) - g \left(\log \left(p_j^{**}(1 - p_j^{**}) \right) + \log(v_j) \right) \right] \\
&+ p_k^{**} \left[g \left(\log \left(p_{jk}^{**}(1 - p_{jk}^{**}) \right) + \log(v_j) \right) - g \left(\log \left(p_k^{**}(1 - p_k^{**}) \right) + \log(v_k) \right) \right] \\
\geq& \, 0,
\end{aligned}
$$

where the inequality follows from $\min\{2p_j^{**} + p_k^{**}, p_j^{**} + 2p_k^{**}\} < 1$, $v_j \geq v_k$, and the monotonicity of $g(\cdot)$. A contradiction. Therefore, only two contestants would remain active in the optimal contest. Moreover, they must be the two ex ante strongest players by Theorem 4.

It remains to show that the ex ante stronger player always wins with a strictly higher probability than the underdog. Suppose, to the contrary, that $v_1 > v_2$ and $0 < p_1^{**} \leq p_2^{**}$, with $p_1^{**} + p_2^{**} = 1$. We consider the following two cases:

Case I: $p_1^{**} < p_2^{**}$**.** Then the designer can increase the expected winner's effort by assigning probability p_1^{**} to player 2 and p_2^{**} to player 1. This would lead to a change in the expected winner's effort that amounts to

$$
\begin{aligned}
&\left[p_1^{**} g \left(\log \left(p_1^{**} p_2^{**} \right) + \log(v_2) \right) + p_2^{**} g \left(\log \left(p_1^{**} p_2^{**} \right) + \log(v_1) \right) \right] \\
&- \left[p_1^{**} g \left(\log \left(p_1^{**} p_2^{**} \right) + \log(v_1) \right) + p_2^{**} g \left(\log \left(p_1^{**} p_2^{**} \right) + \log(v_2) \right) \right] \\
=& \, (p_2^{**} - p_1^{**}) \left[g \left(\log \left(p_1^{**} p_2^{**} \right) + \log(v_1) \right) - g \left(\log \left(p_1^{**} p_2^{**} \right) + \log(v_2) \right) \right] > 0,
\end{aligned}
$$

which is a contradiction.

Case II: $p_1^{**} = p_2^{**} = \frac{1}{2}$**.** Let the designer assign winning probability $\frac{1}{2} + \epsilon$ to player 1 and $\frac{1}{2} - \epsilon$ to player 2. The adjustment leads to a change in the expected winner's effort that amounts to

$$\Xi(\epsilon) := \left[\left(\frac{1}{2}+\epsilon\right) g\left(\log\left(\frac{1}{4}-\epsilon^2\right)+\log(v_1)\right) + \left(\frac{1}{2}-\epsilon\right) g\left(\log\left(\frac{1}{4}-\epsilon^2\right)+\log(v_2)\right)\right] - \frac{1}{2}\left[g\left(\log\left(\frac{1}{4}\right)+\log(v_1)\right) + g\left(\log\left(\frac{1}{4}\right)+\log(v_2)\right)\right].$$

It is straightforward to verify that $\Xi(0) = 0$ and $\Xi'(0) = g\left(\log\left(\frac{v_1}{4}\right)\right) - g\left(\log\left(\frac{v_2}{4}\right)\right) > 0$. Therefore, $\Xi(\epsilon) > 0$ for sufficiently small $\epsilon > 0$, which is again a contradiction. This completes the proof. ■

Proof of Theorem 8

Proof. Recall that Theorem 6 states that $p_i^*, p_j^* < \frac{1}{2}$, $\forall\, i, j \in \mathcal{N}$. Suppose, to the contrary, that $v_i > v_j$ and $p_i^* \leq p_j^*$. We consider the following two cases:

Case I: $p_i^* < p_j^*$. Let the contest designer assign probability p_j^* to player i and p_i^* to player j, and not change the equilibrium winning probability of all other players. Define $\Omega_{k_1k_2} := \log(p_{k_1}^*\left(1-p_{k_1}^*)\right) + \log(v_{k_2})$ for $k_1, k_2 \in \{i, j\}$. It can be verified that $\Omega_{ii}, \Omega_{jj} \in (\Omega_{ij}, \Omega_{ji})$ and $\Omega_{ii} + \Omega_{jj} = \Omega_{ij} + \Omega_{ji}$. Furthermore, the difference between the total effort under the alternative profile of winning probabilities and that under $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ is equal to

$$\left[g\left(\Omega_{ij}\right) + g\left(\Omega_{ji}\right)\right] - \left[g\left(\Omega_{ii}\right) + g\left(\Omega_{jj}\right)\right] > 0,$$

where the strict inequality follows from $\Omega_{ii}, \Omega_{jj} \in (\Omega_{ij}, \Omega_{ji})$, $\Omega_{ii} + \Omega_{jj} = \Omega_{ij} + \Omega_{ji}$, and the strict convexity of $g(\cdot)$. A contradiction.

Case II: $p_i^* = p_j^*$. Let the contest designer assign probability $p_i^* + \epsilon$ to player i and $p_j^* - \epsilon$ to player j, and not change the equilibrium winning probability of all other players. It can be verified that such adjustment generates strictly more total effort to the designer for a sufficiently small $\epsilon > 0$. This completes the proof. ■

Proof of Theorem 10

Proof. Part (ii) of the theorem follows directly from part (i), and it suffices to prove part (i). With slight abuse of notation, we add r into α_i and α_j to emphasize the fact that the optimal weights $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ depend on the bidding efficiency r. Note that $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ and κ are independent of r by Theorem 9. Moreover, we have that

$$\mathcal{T}(r) := \log\left(\frac{\alpha_i^*(r)}{\alpha_j^*(r)}\right) = (1-r)\log\left(\frac{p_i^*}{p_j^*}\right) - r\log\left(\frac{1-p_i^*}{1-p_j^*}\right) - r\log\left(\frac{v_i}{v_j}\right).$$

Clearly, $\mathcal{T}(r)$ is linear in r, and $\mathcal{T}(r) \gtrless 0$ is equivalent to $\alpha_i^*(r) \gtrless \alpha_j^*(r)$. Note that

$$\lim_{r \searrow 0} \mathcal{T}(r) = \log\left(\frac{p_i^*}{p_j^*}\right) > 0,$$

and

$$\mathcal{T}(1) = -\log\left(\frac{1-p_i^*}{1-p_j^*} \times \frac{v_i}{v_j}\right) = -\log\left(\frac{v_i + \frac{\kappa-2}{\sum_{s=1}^{\kappa} \frac{1}{v_s}}}{v_j + \frac{\kappa-2}{\sum_{s=1}^{\kappa} \frac{1}{v_s}}}\right) < 0,$$

where the second equality follows from Equation (12). Therefore, there exists a unique cutoff $\overline{r}_{ij} \in (0,1)$ such that $\alpha_i^*(r) \gtrless \alpha_j^*(r)$ if $r \lesseqgtr \overline{r}_{ij}$. This completes the proof. ■