Disappointment Aversion and Long-Term Dynamic Asset Allocation*

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Abstract

We examine the impact of return predictability and parameter uncertainty on investors' long-term portfolio allocations in the context of disappointment aversion. We find persisting horizon effects, with stocks appearing progressively more attractive at longer horizons as opposed to shorter ones. We find a level of disappointment aversion below which it is optimal for investors to hold zero units of a risky asset. Our analysis has implications for the nonparticipation puzzle in the stock markets.

JEL classification: G11; G12; G40

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1 Introduction

Disappointment constitutes one's emotional response when one's perceived outcomes about the state of the world fall below expectations (i.e., negative outcomes). Given the negative emotional connotations associated with disappointment, people tend to prefer to avoid it, and disappointment aversion (DA, hereafter) has been found to affect investors' asset allocation decisions (see Gul, 1991; Ang et al., 2005; Fielding and Stracca, 2007; Dahlquist et al., 2017). The fact, however, that investment horizons vary across investors (their outlook may be short or long term) indicates that the frequency with which investors experience disappointment (and their aversion to it) is expected to vary over time as well. For example, an investor with a short (e.g., weekly or monthly) horizon would likely witness their strategy underperform their expectations (and, hence, feel disappointed) more often compared with an investor with a longer (e.g., annual) horizon, considering the relatively higher volatility typifying returns in the short run (Benartzi and Thaler, 1995). Although this would suggest the presence of differential DA effects over investors' asset allocation decisions across different horizons, little research to date has explored DA in an intertemporal setting.

This paper aims to address this gap in the literature by first revisiting the single-period version of DA utility by Ang et al. (2005), before formalizing a multi-period dynamic framework in partial equilibrium,[1] which allows for sequential investing and reallocation of the available wealth. Doing so would allow one to investigate how DA preferences affect the decision-making of people who seek to maximize their expected utility of wealth. Investors choose portfolio allocations for two investable assets, a risk-free bond and a risky asset. Because it is impossible to, a priori, assert investors' beliefs as per the return-generation process in the market, we follow the relevant literature by allowing for two possible data-generating processes (DGPs). The first one assumes returns are independent and identically distributed (i.i.d.), and the second one assumes a VAR form that uses the dividend price ratio (i.e., the dividend yield) as its predictor variable.

[1]Our approach employs an exogenous price setting, making this study a partial equilibrium one. Modeling the cash flows can lead to an endogenous price setting, where equilibrium asset prices are attained and markets clear. For an example of an equilibrium study, see Lynch (2000).

Our study presents evidence strongly supporting the role of DA in defining equity participation (and nonparticipation) regions. We show, both mathematically and empirically, that, for every portfolio allocation and level of expected equity return, there is a critical value of DA below which it is optimal for a DA investor to hold zero units of the risky asset. More interestingly, we find that DA investors tend to allocate significantly less to equity compared to investors with isoelastic (power) utility. DA appears to be powerful at every horizon: we find that a small increase of the DA coefficient leads to a significant decrease in equity holdings in the case of an investor who accounts for predictability in stock returns. A buy-and-hold DA investor, as opposed to one who follows a dynamic strategy, increases the allocation to the risky asset with investment horizon regardless of whether she considers predictability in the returns. The former, assuming returns are predictable, will decrease her allocation amount to the risky asset for both long and short horizons. Again, the impact of the DA coefficient is drastic: the more disappointment averse the investor becomes, the less the portfolio weight she assigns to equity.

An issue arising from the above two DGPs pertains to the uncertainty inherent in their estimated parameters and how investors may treat it. To assess the impact of parameter uncertainty, we compare asset allocations to the risky asset between cases in which parameter uncertainty is ignored and others where it is considered, for both DGPs. We observe the rise of differential horizon effects when parameter uncertainty is either ignored or incorporated for investors who either use the i.i.d. return-generating process or account for return predictability. A moderately risk-averse, buy-and-hold investor will allocate a large part of her wealth to the risky asset for long (as opposed to short) horizons when parameter uncertainty is ignored, taking advantage of the lower per-period volatility of the risky asset's returns, which, in turn, decreases the cumulative volatility she experiences over the investment horizon. When the investor considers parameter uncertainty in her investment strategy, she will again allocate larger portfolio weights to the risky asset for longer horizons, however, the impact of parameter uncertainty is clear: Incorporating it in the asset allocation exercise reduces significantly the exposure of the portfolio to the risky asset, compared to a strategy indifferent to parameter uncertainty, especially at very long horizons (i.e., longer than twenty years), thus preventing overallocation to equities.

Our paper produces a series of original contributions to the extant literature on investors' DA portfolio choices contextualized by predictability and parameter uncertainty. First, we extend the study of investors' portfolio choices with DA utility by providing optimal participation conditions and nonparticipation regions both for static (buy-and-hold) and dynamic allocations. Second, we revisit and extend the study of the portfolio choice problem for a long-term buy-and-hold investor under return predictability and parameter uncertainty. Although our study primarily focuses on dynamic portfolio choice, revisiting the buy-and-hold asset allocation problem for very long investment horizons (up to 40 years) reveals a number of important implications for the different investment behaviors of a long versus a short-term DA investor. To our knowledge, this specific focus has not attracted attention so far.[2] Third, we demonstrate how the incorporation of predictability into asset returns affects portfolio weights at different horizons for a dynamic investor and how this can give rise to horizon effects, in the sense that investors change their portfolio's composition taking into account the variability in investment opportunities. Finally, we complete our study constructing a Bayesian framework that incorporates both predictability and parameter uncertainty to investigate how each of the two properties affect portfolio compositions in a DA context. Here, the choice of the risky asset return generator is crucial; for example, the impact of parameter uncertainty on a dynamic strategy, where returns are i.i.d., is not as powerful as that in the case in which predictability is considered, leading to significantly different portfolio allocations.

2 Literature Review

Our research is primarily motivated by extant evidence, according to which investors do not strictly adhere to the assumptions of expected utility theory in their

[2]Dynamic portfolio allocation has, overall, been widely studied, in both discrete time and continuous time (Campbell and Viceira, 2002; Brandt et al., 2005; Aït-Sahalia et al., 2009).

decisions,[3] instead being prone to viewing choices in a biased fashion, often under the influence of emotions and cognitive biases (such as mental accounting and framing effects).[4] Several theoretical propositions (Handa, 1977; Chew and MacCrimmon, 1979; Quiggin, 1982; Fishburn, 1983; Tversky and Kahneman, 1992) depart from the expected utility framework to reflect more accurately on investors' decision-making under risk, transforming probabilities into decision weights through nonlinear probability functions. Prospect theory (PT, hereafter), in particular, has proved particularly successful in capturing frequently encountered traits of investors' behavior (Kahneman and Tversky, 1979; Berkelaar et al., 2004; Gomes, 2005; Barberis and Huang, 2008; Dimmock and Kouwenberg, 2010; Bernard and Ghossoub, 2010). In prospect theory, investors are assumed to evaluate the performance of their investments by anchoring to some historical reference point, engaging in the computation of gains and losses relative to that point. Investors also respond asymmetrically to gains versus losses—courtesy of loss aversion—by exhibiting greater sensitivity to losses compared to gains, a sensitivity further reflected in the PT value function, which grows steeper in the loss region. As a result, investors seek more risk when in the domain of losses (they hold onto their loser stocks hoping for a price rebound) and are more risk averse in the domain of gains (they sell their winner stocks to realize profits, while profits still exist).[5]

A derivative of PT is DA theory, formally introduced by Gul (1991).[6] DA theory

[3]In practice, for example, the independence axiom is frequently violated, with the Allais paradox representing the most famous evidence of the latter. Other notable violations of the expected utility framework are observed in Ellsberg's paradox and the St. Petersburg paradox. For more on those, see Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979), and Andreoni and Sprenger (2010).

[4]In the case of mental accounting, investors holding a portfolio of stocks may treat the performance of each stock in isolation, instead of viewing the stock as part of the portfolio (Barberis et al., 2001). Framing, on the other hand, can lead investors to choose an option that appears attractive on the background of less-attractive alternatives and not because it is the optimal option. For a more detailed discussion of the above, see Kahneman et al. (2011).

[5]This in essence leads investors to sell their winning assets more quickly compared with their losing ones. Empirical evidence (Odean, 1998; Grinblatt and Keloharju, 2001; Locke and Mann, 2001; Shapira and Venezia, 2001; Wermers, 2003; Haigh and List, 2005; Jin and Scherbina, 2010) suggests that this pattern permeates both retail and institutional investors' behavior internationally, yet leads to suboptimal performance. The latter has been ascribed to the effect of short-term momentum (Jegadeesh and Titman, 1993, 2001) in stock returns, according to which recent winners (losers) will continue outperforming (underperforming) in the near future; this finding, in turn, suggests that investors in the prospect theory setting should keep their winners (instead of quickly selling them) and sell their losers (instead of keeping them). A potential explanation for the momentum effect can be given in the context of a realization utility model (Barberis and Xiong, 2012), where selling well-performing stocks to realize gains is attributed to the additional utility derived by real instead of paper profits.

[6]Although this framework is recognized as the DA theory one, Bell (1985) first studied the disappoint-

extends the expected utility theory by relaxing the independence axiom, while retaining the basic features of PT (asymmetric preferences, reference dependence, diminishing sensitivity, and probability weighting). Moreover, it provides us with better understanding in the way the certainty equivalent of wealth is chosen and updated. The certainty equivalent of wealth represents the certain level of wealth W that generates the same level of utility as a portfolio composition that yields a (noncertain) wealth level W. In a DA context, the certainty equivalent of wealth serves as a reference point for investor's wealth against which gains and losses are compared. In PT these points are set exogenously and are usually equal to the current wealth (the status quo), whereas in DA theory they are updated in an endogenous way.

In terms of portfolio choice, Ang et al. (2005) integrate the DA theory with asset allocation in a dynamic programming setup. They primarily address the portfolio choice problem in a single-period setting, with DA utility investors allocating their wealth between a risk-free security and a risky asset. They find that incorporating DA leads to more plausible portfolio compositions with a smaller proportion of wealth allocated to the risky asset. Doing so provide a reasonable explanation for the observed equity premium (and concomitant nonparticipation) puzzle. As a result, DA is relevant to an individual's decision-making and should be taken into consideration in asset allocation. We use the same wealth utility function used by Ang et al. (2005) to revisit and complete the single-period asset allocation problem. Our focus is to study the multi-period dynamic problem, and, in particular, we are interested in the effects of return predictability and parameter uncertainty on investors' portfolio composition.

By deriving the optimal portfolio comprising a risky and a risk-free asset for an investor who uses the DA utility function, we extend the DA-related literature. Overall, empirical applications of the DA theory have been rather limited to date, a fact attributed by Abdellaoui and Bleichrodt (2007) to the theory lacking a method of formally extracting the DA coefficient. To that end, Abdellaoui and Bleichrodt (2007) proposed a trade-off method, which first derives the underlying

ment effect arising from the discrepancy between an agent's prior expectations and realized outcomes.

utility function and then, based on the function, extracts the DA coefficient. DA theory has been mainly used in asset pricing settings (Routledge and Zin, 2010; Bonomo et al., 2011), where a slightly altered version of the original DA theory is used. More specifically, these studies consider a generalized version of Gul (1991)'s framework, extending the DA utility by an additional term and a new coefficient on top of the DA coefficient. In these setups, an outcome signals "disappointment" only when it lies sufficiently below the certainty equivalent, as determined by the additional coefficient and the DA parameter. In asset allocation setups, Dahlquist et al. (2017) employed DA preferences to derive analytical expressions for measures such as the effective risk aversion when studying higher moments of return distributions.

Finally, we contribute to the growing portfolio choice literature, which discusses the incorporation of parameter uncertainty into the asset allocation issue. Relevant literature (Bawa et al., 1979; Kandel and Stambaugh, 1996; Barberis, 2000; Avramov and Zhou, 2010; Kacperczyk and Damien, 2011) integrates several forms of uncertainty (model, parameter, or distribution) with asset allocation decision-making. Recently obtained evidence further corroborates the importance of predictability and parameter uncertainty for portfolio choices. Branger et al. (2013) and DeMiguel et al. (2015) examine the construction of optimal portfolios under uncertainty about expected asset returns and find that parameter uncertainty is highly relevant to portfolio choice. Chen et al. (2014) study the dynamic portfolio choice problem when investors face uncertainty about the model's specification, incorporating learning to construct strategies that depart from the Bayesian approach. Hoevenaars et al. (2014) test the impact of different uninformative priors on both short- and long-term equity allocations, whereas Johannes et al. (2014) investigate the impact of predictability and parameter uncertainty in an expected utility framework mainly focusing on the impact of volatility on the portfolio choice problem.

3 DA Framework

We define the DA framework employed in this study as follows:

$$U(\mu_W) = \frac{1}{K}\left(\int_{-\infty}^{\mu_W} U(W)dF(W) + A \int_{\mu_W}^{\infty} U(W)dF(W) \right), \tag{1}$$

where A is the coefficient of DA, bounded between zero and one (i.e., $0 < A \leq 1$); $U(\cdot)$ is the constant relative risk aversion (CRRA) utility function defined by $U(W) = W^{1-\gamma}/(1-\gamma)$; μ_W is the certainty equivalent of wealth; $F(\cdot)$ is the cumulative distribution function for wealth; and K is a scalar equal to $P(W \leq \mu_W) + AP(W > \mu_W)$. Assume two assets, one risky asset and one risk-free asset, whose continuously compounded returns are denoted by e^y and e^r, respectively. Then the investor's wealth is defined as $W = \alpha X + e^r$, where α is investment in the risky asset as a percentage of the investor's total investment (i.e., the weight of the risky asset); $X = e^y - e^r$ is the excess risky asset's return; and the initial wealth is set to one, because the optimization problem is homogeneous in wealth under the CRRA utility function. When a DA investor allocates her wealth into assets in order to maximize the DA utility for a single period, the static optimization problem is

$$\max_{\alpha} U(\mu_W). \tag{2}$$

The above constitutes the asset allocation problem under the assumption of DA utility in a single-period setting.

3.1 Dynamic allocation with DA utility

A dynamic optimization problem with DA utility in a multiple-period setting is considerably more complicated than in a static one, because at every horizon the optimization routine should take into account the investment opportunity set for the whole remaining investment period (as opposed to a one-period forward-looking myopic strategy), while the certainty equivalent of wealth is itself a function of each step's optimal decision. The complexity of the optimization problem further increases by considering predictability, which leads to stochastic investment opportunity sets. We begin by first analyzing a conventional utility function defined over wealth $U(W)$ and then moving to the dynamic asset allocation with DA utility.

3.1.1 Dynamic asset allocation with general utility of wealth.

Assume the following dynamic asset allocation problem in discrete time, where an agent aims to maximize the expected utility of the end-of-period wealth W_T as follows:

$$\max_{\alpha_0,\alpha_1,\ldots,\alpha_{T-1}} \mathbb{E}_0[U(W_T)], \tag{3}$$

where $\alpha_0, \alpha_1, ..., \alpha_{T-1}$ are the investment proportions of the risky asset at times $t = 0, 1, \ldots, T-1$, respectively, and $U(W) = W^{1-\gamma}/1-\gamma$. In this problem, the investor allocates her wealth at time $t = 0$ for T periods, at $t = 1$ for $T-1$ periods and so on until she reaches time $t = T-1$, where she invests for a single period.[7] Wealth W_{t+1} is defined as $W_{t+1} = W_t R_{t+1}(\alpha_t)$, where $R_{t+1}(\alpha_t)$ and α_t represent the total portfolio return over the period t to $t+1$ and the investment weight on the risky asset at time t, respectively. At time t when the investor seeks to allocate her available wealth optimally between the risky and the riskless asset in order to maximize her expected utility, the optimization problem becomes

$$\max_{\alpha_t} \mathbb{E}_t[U(W_{t+1}Q^*_{t+1,T})], \tag{4}$$

where $Q^*_{t+1,T} = R_T(\alpha^*_{T-1})R_{T-1}(\alpha^*_{T-2})\cdots R_{t+2}(\alpha^*_{t+1})$ represents the aggregate return from time $t+1$ to T that maximizes the investor's expected utility.

Using dynamic programming, we solve the problem at time $t = T-1$ for the asset allocation decision for the period $T-1$ to T. Continuing recursively, we solve the asset allocation subproblem at time $T-2$ using the solution to the problem at $T-1$, until we reach time t. This procedure derives a final solution for the portfolio allocation to the risky asset $\alpha_t, \alpha_{t+1}, \ldots, \alpha_{T-1}$ that will be optimal as guaranteed by the principle of optimality in dynamic programming.[8] For the power utility

[7]This problem mimics the optimization problem that pension fund managers face over multiple periods.

[8]See Bertsekas (1995) for more details on that.

function, the expression in (4) takes the form of

$$\max_{\alpha_t} \mathbb{E}_t \left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma} (Q^*_{t+1,T})^{1-\gamma} \right]. \tag{5}$$

Backward induction suggests that $Q_{t+1,T}$ is optimal, because it represents the optimal investment decision between times $t+1$ and T that maximizes the expected utility. We calculate the optimal investment proportions of the risky asset at every time step of the investment period as

$$\alpha^*_t = \arg\max_{\alpha_t} \mathbb{E}_t \left[W_{t+1}^{1-\gamma} (Q^*_{t+1,T})^{1-\gamma} \right]. \tag{6}$$

3.1.2 Dynamic asset allocation with DA utility.

DA utility incorporates CRRA preferences as a special case in which $A = 1$, but the dynamic extension of the single-period problem for DA utility is far more complicated because of the so-called "curse of dimensionality", i.e., the number of state variables exponentially increases with time.[9] We begin by first formulating the dynamic optimization problem between t and T.

Proposition 1 *For given $Q^*_{t+1,T} = R_T(\alpha^*_{T-1}) R_{T-1}(\alpha^*_{T-2}) \cdots R_{t+2}(\alpha^*_{t+1})$, the DA utility function for the dynamic asset allocation problem is given by*

$$\begin{aligned} U(\mu_t) = \frac{1}{K_t} \Bigg[&\mathbb{E}_t \Big(U(W_{t+1} Q^*_{t+1,T}) \mathbf{1}_{W_{t+1} Q^*_{t+1,T} \le \mu_t} \Big) \\ &+ A \mathbb{E}_t \Big(U(W_{t+1} Q^*_{t+1,T}) \mathbf{1}_{W_{t+1} Q^*_{t+1,T} > \mu_t} \Big) \Bigg], \end{aligned} \tag{7}$$

where $W_{t+1} Q^*_{t+1,T} = W_T$, according to the recursive definition of wealth. The first-order condition (FOC) for optimizing the utility of the certainty equivalent return

[9] As the state variables take a number of different values at each horizon, the state-space exponentially increases with time with every iteration of the algorithm. For example, a T-period problem with a state variable with s states produces s^T possible combinations. From an analytical perspective, this is not a big obstacle (as the problem still can be mathematically formulated), but computation-wise, the exponential increment of the state-space renders the use of algorithmic processes problematic.

is given by

$$\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\leq\mu_t}\right) + A\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\right) = 0, \quad (8)$$

where $X_{t+1} = e^{y_{t+1}} - e^{r_t}$ *is the excess return of the risky asset over the riskless asset.*

Proof. See Appendix C.1

The main drawback with Proposition 1 is that recursive optimization exponentially increases the state space in $Q_{t+1,T}$ in order to take into account all the possible states for the return of the risky asset between times $t+1$ and T. To overcome the "curse of dimensionality," we elaborate on the approach proposed in Epstein and Zin (1989) and Ang et al. (2005), by assuming that future uncertainty about the risky asset's returns is captured by the certainty equivalent. Under this approach, instead of carrying backward all the possible states for the equity return at each horizon, we pay attention to only one variable, the next-period's certainty equivalent, keeping the dimension of the state space to the minimum possible. Let μ_t represent the certainty equivalent return for the utility at time $t+1$ with the optimal asset allocation:

$$\max_{\alpha_t}\mathbb{E}(U(W_{t+1})) = \max_{\alpha_t}U(W_t\mu_t(\alpha_t)). \quad (9)$$

Then we obtain the following result:

Proposition 2 *The utility of the certainty equivalent return at time* $0 \leq t < T-1$ *is as follows:*

$$U(\mu_t) = \frac{1}{K_t}\left[\mathbb{E}_t\left(U(R_{t+1}(\alpha_t)W_t\prod_{i=t+1}^{T-1}\mu_i^*)\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\xi_t\}}\right) + A\mathbb{E}_t\left(U(R_{t+1}(\alpha_t)W_t\prod_{i=t+1}^{T-1}\mu_i^*)\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}}\right)\right]. \quad (10)$$

The value of $U(\mu_t)$ for the boundary condition $t = T-1$ is given by

$$U(\mu_{T-1}) = \frac{1}{K_{T-1}}\Big[\mathbb{E}_{T-1}\big(U(R_T(\alpha_{T-1})W_{T-1})\mathbf{1}_{\{R_T(\alpha_{T-1})\leq\mu_{T-1}\}}\big) \\ + A\mathbb{E}_{T-1}\big(U(R_T(\alpha_{T-1})W_{T-1})\mathbf{1}_{\{R_T(\alpha_{T-1})>\mu_{T-1}\}}\big)\Big], \quad (11)$$

and the FOC for optimizing the utility of the certainty equivalent return is given by

$$\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\xi_t\}}\right)+A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}}\right) = 0, \quad (12)$$

where $\xi_t = \frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}$, with μ^*'s as the optimal certainty equivalents between $t+1$ and $T-1$.

Proof. See Appendix C.2.

Remark *Notice that W_t will eventually not be part of the expressions for $U(\mu_t)$ in Equations (10) and (11) as moving backward in time we will have $W_t = W_0\prod_{i=1}^{t} R_i(\alpha_{i-1})$, where all the uncertainty about $R_n(\alpha_{n-1})$, where $n \in \{1, 2, \ldots, T\}$, will be captured by the certainty equivalent return μ^*_n, where $n \in \{1, 2, \ldots, T\}$, and W_0 is set to one given that the optimization problem is homogeneous for wealth.*

We also notice that the investor's gains or losses at time $t+1$ are now calculated with respect to ξ_t, that is, the certainty equivalent at time t for the optimal certainty equivalent from $t+1$ to T. By substituting portfolio returns with the corresponding certainty equivalent, the state-space comprises a constant number of states that remain constant with time. As an example of the advantage of using the certainty equivalent, we can present the FOC in Equation (12) for power utility as follows:

$$\mathbb{E}_t\big(R^{-\gamma}_{t+1}(\alpha_t)X_{t+1}\mathbf{1}_{R_{t+1(\alpha_t)\leq\xi_t}}\big) + A\mathbb{E}_t\big(R^{-\gamma}_{t+1}(\alpha_t)X_{t+1}\mathbf{1}_{R_{t+1(\alpha_t)>\xi_t}}\big) = 0. \quad (13)$$

To determine the optimal numerical values for μ_t and α_t, we adopt a Gaussian quadrature scheme (see Davis and Rabinowitz, 2007, for an in-depth review of numerical integration methods) like in Balduzzi and Lynch (1999) and Campbell and Viceira (1999) to track the states $\{R^s_{t+1}(\alpha_t)\}_{s=1}^{N}\big(\prod_{i=t+1}^{T-1}\mu^*_i\big)$, where N is the number of quadrature states for the equity return.[10] Next, we solve the discretized expression of Equation (10) (adjusted for power utility) in parallel with the FOC for the DA maximization problem in Equation (13) like in the static single-period case, but recursively incorporating the calculations from periods $T-1$ to $t+1$. Appendix B offers details about the discretization of the DA allocation problem and its solution.

3.2 Nonparticipation under DA utility

The case for nonparticipation in risky assets has been the subject of considerable research to date. Mental accounting (which assumes the nonfungibility of monetary resources allocated to each asset; see e.g., Thaler and Sunstein, 2008) motivates narrow framing (Barberis and Huang, 2008), which prompts investors to perceive high-volatility assets as "risky" in isolation without assessing their contributions to the risk-return profile of a portfolio. Nonparticipation also can be promoted by the omission bias (Ritov and Baron, 1999), whereby omissions (e.g., not investing in stocks) are favored over equivalent commissions (investing in stocks), because commissions, unlike omissions, involve commitment to a course of action, thus entailing the possibility of a loss. Other alternative explanations proposed to account for nonparticipation include the familiarity bias (choosing more over less familiar assets, believing the latter to be riskier; Huberman, 2001; Massa and Simonov, 2006), the recognition bias (preferring more over less recognizable assets; Boyd, 2001) and limited cognition (when investors view risk diversification as a decision of enhanced complexity; Hirshleifer, 2008).

3.2.1 Single period

Under CRRA preferences holding positive portfolio weights for risky assets when the expected excess return is positive ($\mathbb{E}(X) > 0$) is always the optimal choice for investors; however, this is not always the case with DA utility preferences.

[10] Instead of quadrature-based methods, Monte-Carlo simulations or, even, regression-based methods, like in Brandt et al. (2005), can be used to calculate the expectations in Equation (13). In practice however, the quadrature method offers sufficient accuracy and greater computational speed compared with the alternatives.

Under DA preferences refraining from holding risky assets even if the expected excess return is positive can be the optimal choice for investors in some cases. This nonparticipation region in the following theorem shows that it is not optimal to hold risky assets whenever the DA coefficient lies below a critical value (A^*).

Theorem 1 *Let* $\mu = \mu_W(A, \alpha)$, *with*

- $\mu(A, :) \in \mathbf{C}^1, \forall A \in [0, 1]$
- $\frac{d\mu(A,0)}{d\alpha} = \xi(A) \leq 0, \forall A \in [0, 1]$[11]
- $\mathbb{E}(X) > 0$ *and* $\mathbb{E}(X\mathbf{1}_{W \geq \xi(A)}) > 0$, *where* $X = e^y - e^r$ *is the return of the risky asset in excess of the risk–free rate.*

Then, setting

$$A^* = \frac{\mathbb{E}(X\mathbf{1}_{W \geq \xi(A)})}{\mathbb{E}(X\mathbf{1}_{W < \xi(A)})}, \tag{14}$$

we have the following:

1. *For every* $A \leq A^*$, $\alpha^* = 0$,
2. *For every* $A > A^*$, $\alpha^* > 0$,

where α^* *is the optimal investment proportion in the risky asset which maximizes* $\mu(A, \alpha)$ *for a given* A. A^* *is independent of the risk aversion parameter* γ.

Proof. See appendix C.3

This theorem can be intuitively presented in the following way: focusing on the DA coefficient A, we find that, as DA increases, investors allocate less wealth to the risky asset regardless of their level of risk aversion. Given that the utility of wealth is a continuous function within the domain of A, there should be a level of A, let A^*, at which the optimal portfolio allocation to the risky asset, α^*, equals zero. This result obtains independently of the level of risk aversion γ. Recalling the condition $d\mu(A, 0)/d\alpha \leq 0$, we see that a further decrease in the portfolio weight allocated to the risky asset α^* (e.g., because of short selling the risky asset) will result in a higher

[11] Positive risk premium when the end-of-period wealth exceeds the negative impact of the decrease in the certainty equivalent as the investment proportion of the risky asset increases. Suppose that the expected return of the risky asset is zero. The certainty equivalent decreases when the proportion of the risky asset increases. This occurs because for $\alpha < 0$ negative excess return states have higher wealth than R and hence are downweighted (Ang et al., 2005).

certainty equivalent. When investment in the risky asset is zero, an increase in the investment in the risky asset decreases the certainty equivalent. This is intuitively correct, because, by increasing the portfolio allocation to equities to a nonzero level, investors become more willing to accept an amount of risk instead of holding only the risk-free security. This consequently implies that the monetary amount that can keep investors away from buying stocks should be lower now. Subsequently, the following relationship will prevail:

$$W = \alpha^* X + R > R,$$

for $\alpha^* < 0$ and negative states ($X < 0$) of the excess equity return. Therefore, the optimal allocation for this critical level of the DA coefficient, A^*, is zero and $\alpha = \alpha^* = 0$.

[Figure 1 about here.]

3.2.2 Multi-period

To calculate A^* at different horizons, we perform a number of Monte Carlo (MC) experiments for the buy-and-hold and dynamic allocation problems, where we simulate several asset return trajectories under the i.i.d. assumption and using the DGP with predictability to estimate the excess return and the corresponding return volatility. Then, using a binary search algorithm, we are able to extract the critical DA coefficient A^* (which results in allocating zero wealth to equity) for each problem.[12] A typical binary search algorithm for our problem proceeds with discretizing the state space of A (i.e., assuming discrete points on the interval $(0, 1]$) and subsequently performing sequential searches for the target value of A (i.e., the one that makes $\alpha = 0$). This is performed by comparing the target value to the middle element of the state space and cutting the state space in half with every iteration until the optimal value is detected (for the implementation details of the binary search algorithm, see sections 3.1 and 3.2 in Sedgewick and Wayne, 2011).

In the numerical experiments, we follow the simple assumption of normality,

[12] An alternative approach employs the calibration of a binary tree and the detection of the correct interval for A^* (Ang et al., 2005). In practice, both methods derive similar results for A^*.

where risky asset returns follow an i.i.d. process, which we also relax by assuming predictable asset returns. To model the predictability of asset returns, we use a vector autoregression (VAR) model, which has been frequently used in the asset allocation literature in discrete time (see Barberis, 2000; Ang et al., 2005; Brandt et al., 2005; Hoevenaars et al., 2014, among others). In our VAR, after examining a number of candidate variables (see Section 2.3.3 for more details), the dividend price ratio $(d/p)_t$ mainly drives the next-period's equity return.

In the MC simulations, quarterly data for the S&P 500 index and the 3-month Treasury bill proxy for the equity returns and the risk-free rate, respectively. Our numerical experiments corroborate Theorem 1, i.e., risk aversion does not affect the nonparticipation region for the DA coefficient.

[Figure 2 about here.]

The left graph of Fig. 2 plots the critical level of the coefficient of DA (A^*) across investment horizons for a buy-and-hold DA investor. The DA coefficient (A^*) is critical, because a DA investor should not hold any units of the risky asset if her DA lies below A^*. A decreasing A^* within these setting results in larger market participation, as a lower A^* implies that the investor has to be more disappointment averse to refrain from holding the risky asset. For a length longer than a 5-year period, a DA investor who follows a buy-and-hold strategy will hold risky assets regardless of the DGP assumed for equity returns.

The right graph of Fig. 2 reports critical levels of A^* for dynamic asset allocation strategies for various investment horizons ($T-t$, where t is the current horizon). In the case of i.i.d. returns (dashed line), the critical DA coefficient remains constant regardless of the investment horizon as a result of the invariable opportunity set. The solid line corresponds to predictable returns using the VAR to forecast the next-period's equity return as a function of the dividend price ratio. Contrary to the case of i.i.d. returns, where A^* remains constant, investors' participation increases at longer horizons.

3.3 Asset allocation with parameter uncertainty

One can investigate the effects of parameter uncertainty on asset allocation by allowing for uncertainty in the parameter estimates. At time t investors maximize the following utility function:

$$\max_{\alpha} \int_{-\infty}^{\infty} U(W_{t+n}) p(r_{t+n}|Y;\theta) dr_{t+n}, \tag{15}$$

where n is the investor's horizon; $U(\cdot)$ is her utility of wealth; and $p(r_{t+n}|Y;\theta)$ is the cumulative density function of the expected returns conditional on observed return data Y and the set of parameters θ (in our case the mean and variance of the risky asset's return). Uncertainty arises for θ, because these parameters become known only after the end of the investment horizon. A popular approach in the literature for maneuvering the parameter uncertainty problem is to use a Bayesian framework that incorporates uncertainty in the parameters of θ. Integrating out θ in the prior distribution $p(r_{t+n}|Y;\theta)$, we obtain the posterior predictive distribution, which updates the distribution parameters by embodying the new data. A DA investor now maximizes

$$\max_{\alpha} \left[\int_{W_{t+n} \leq \mu_W} U(W_{t+n}) p(r_{t+n}|Y) dr_{t+n} + A \int_{W_{t+n} > \mu_W} U(W_{t+n}) p(r_{t+n}|Y) dr_{t+n} \right], \tag{16}$$

in place of Equation (15), in line with the DA utility definition in Equation (1), where the distribution of the returns is now conditional on observed stock return data only, not on the set θ.

3.3.1 Data

To study the problem of portfolio choice, we utilize quarterly data from the U.S. market from January 1934 to September 2016 for the S&P 500 index (index returns), the 3-month Treasury bill rate (which represents our risk-free asset), and the dividend price ratio (dividend yield). The latter is the predictor variable for the empirical part of this work. To calculate the annual dividend price ratio, we sum all dividends paid throughout each year and divide them by the year-end index level of the S&P 500. The data sets related to the S&P 500 returns and the 3-month T-bill rates can be easily acquired by a number of sources as they are readily available online.[13]

3.3.2 i.i.d. returns.

When investors ignore predictability in returns, they consider them to be i.i.d., and they use the following model to estimate the next-period's excess equity return:

$$x_t = (\mu - r) + \epsilon_t, \tag{17}$$

where x_t is the continuously compounded quarterly excess return of the S&P 500 index in quarter t, and ϵ_t are i.i.d. disturbance terms distributed as $\epsilon_t \sim \mathcal{N}(0, \sigma)$. The parameter values in Equation (17) are $\mu = 0.02515$, $r = 0.00854$, and $\sigma = 0.08175$, all of which are given in Table 1.

[Table 1 about here.]

Assuming investors are unaware of the true parameter value, we use an uninformative (diffuse) prior of the following type:

$$p(\mu, \sigma) d\mu d\sigma \propto \frac{1}{\sigma} d\mu d\sigma, \tag{18}$$

whereas the joint posterior of the mean return μ and volatility σ is

$$p(\mu, \sigma | Y) \propto p(\mu, \sigma) \times L(\mu, \sigma | Y), \tag{19}$$

where L is the likelihood function. The following lemmas report the results for the case of i.i.d. returns (Lemma 1) and predictive returns (Lemma 2), where the VAR is used.

Lemma 1 *The distribution of the posterior moments for the case of i.i.d. returns*

[13] Our sources are the online platform of Bloomberg Professional Services (for the data on S&P 500 returns) and the Federal Reserve (for the interest rate).

is given by

$$\sigma^2|Y \sim Inv - Gamma\left(\frac{N}{2}, \frac{1}{2}\sum_{i=1}^{N+1}(y_i - \overline{\mu})^2\right)$$
$$\mu|\sigma, Y \sim \mathcal{N}\left(\overline{\mu}, \frac{\sigma^2}{N}\right),$$

where Y is the observed asset return data; N is the sample size; and $\overline{\mu}$ is the sample mean.

Proof. See Appendix D.1.

To construct the posterior predictive distribution for the i.i.d. returns of the risky asset, we follow a standard sampling technique. We first sample once from the marginal posterior distribution $p(\sigma^2|Y)$, and then we use the draw for σ to sample from the posterior distribution $p(\mu|\sigma, Y)$, which is now conditional on σ. We repeat this process to generate a sufficient number (i.e., 1,000,000) of pairs (μ, σ) to create return values and subsequently the posterior distribution for the returns of the risky asset, by sampling once for each pair (μ, σ). Appendix D.1 provides details about the sampling procedure from the derived distributions for the mean and variance.

3.3.3 Return Predictability

In practice, asset returns do not follow a random walk. Researchers have documented factors that can be used to predict part of the variability in asset returns (Lettau and Ludvigson, 2001; Campbell and Yogo, 2006; Ang and Bekaert, 2007; Cochrane, 2008). Investors use available information to predict future returns for optimal asset allocation problems. In our study, we replicate the prediction process using a VAR model, where asset returns and the predictable variable are jointly considered.

This results in time-varying investment opportunity sets, which are conditional on the predictor variable in the VAR model. Instinctively, investors modify the proportion of their current investment allocated to the risky asset. We examined a number of financial variables,[14] and chose the dividend yield (calculated as the

[14]To determine the variable that best fits our data, we test the following predictors: dividend yield (the sum of the dividends over a year divided by the level of the index at the end of the year); term spread

dividend price ratio for the S&P 500 Composite Index) to drive the next-period's equity return. The optimal lag length was calculated as one lag, as confirmed by both the Akaike and the Bayesian information criteria. We then model the dividend-adjusted log excess returns of the risky asset as a first-order VAR of the following form:

$$X_t = C + BX_{t-1} + E_t. \tag{20}$$

In the model of Equation (20) $X_t = \begin{pmatrix} y_t - r_{t-1} \\ (d/p)_{t-1} \end{pmatrix}$ where $y_t - r_{t-1} = x_t$ is the excess equity return; r_t is the risk-free rate; $(d/p)_{t-1}$ is the dividend price ratio; B is the (2 $\times$ 2) matrix of the autoregression coefficients; C is a (2×1) vector of the constant terms; and E is a vector of $i.i.d.$ normally distributed disturbance terms.

We use the lagged rate r_{t-1} to indicate that the value of the risk-free rate is known at the time of portfolio formation $t-1$, in contrast to the risky asset, whose return becomes known at time t only. When asset returns are not predictable, all elements of the matrix with the autoregressive coefficients B are not different from zero, and returns are assumed to be i.i.d. As a result, the VAR model reduces to the i.i.d. return generator of Equation (17). We utilize maximum likelihood estimation (MLE) to calculate the VAR in Equation (20), and Table 2 reports the results.

[Table 2 about here.]

Simulating return trajectories under the assumption that the dividend yield at time t can forecast asset returns at time $t+1$, we match the first two moments of the historical returns' distribution up to two to three significant figures. All coefficients of the matrix with the autoregressive parameters B are statistically significant at the 5% level, and both series (dividend yield and excess asset log returns) are stationary.

(the difference between the 10-year Treasury bond and the 1-year Treasury bond); credit spread (the difference between Moody's BAA corporate bond yield and its AAA equivalent); the 3-month Treasury bill; and the 10-year Treasury bond. The criteria for selecting the best fit are (a) whether a variable enters the VAR as statistically significant and (b) how much of the risky asset's excess return variability it explains.

3.3.4 Parameter Uncertainty with Return Predictability

The VAR in Equation (20) also can be written in its compact form:

$$X = BZ + E, \tag{21}$$

where $X = (X_1 \dots X_T)$ is a $(2 \times T)$ matrix with the number of observations T for the estimated variables; $Z = (z_0 \dots z_T)$ a $(3 \times T)$ matrix; $B \equiv (c\, B)$ is a (2×3) matrix of the autoregressive coefficients and the constant terms; and the $E = (\epsilon_1 \dots \epsilon_T)$ is a $(2 \times T)$ matrix with the uncorrelated disturbance terms. A suitable uninformative prior is the Jeffreys prior given by

$$p(B, \Sigma) = p(B)p(\Sigma) \propto |\Sigma|^{-(m+1)/2}, \tag{22}$$

where $m = 2$ is the total number of regressors on the left-hand side of Equation (21); $p(B)$ is constant; and B is independent of Σ. We obtain the posterior density for the parameter matrix B and the covariance matrix of Equation (21) by the following lemma.

Lemma 2 *The posterior distribution, $p(vec(B), \Sigma|X)$ for the coefficient matrix, B and the variance-covariance matrix, Σ conditional on data X is given by*

$$\Sigma|X \sim \mathcal{W}^{-1}((X - Z\hat{B})'((X - Z\hat{B}), T - n - 1)$$
$$vec(B)|\Sigma, X \sim \mathcal{N}(vec(\hat{B}), \Sigma^{-1}Z'Z),$$

where T is the number of observations in our sample, and n is the number of predictor variables.

Proof. See Appendix (D.2).

Again, to sample from $p(vec(B), \Sigma|X)$, we sample first from $p(\Sigma|X)$—the variance - covariance matrix—conditional on data set X and then, given this draw, from the posterior distribution $p(vec(B), \Sigma|X)$, which will give a draw for the matrix of the VAR coefficients. Appendix (D.2) presents the details of this process and the return-generating procedure.

4 Asset Allocation with DA Preference

4.1 Buy-and-hold strategies

We first investigate the asset allocation problem for different investment horizons for buy-and-hold strategies. Here, agents choose a static portfolio allocation strategy at the beginning of the investment horizon without optimal reallocation. This strategy results in the same allocation regardless of the investment horizon for an investor with power utility when returns follow an i.i.d. process (i.e., are normally distributed). Our goal here is to explore the effects of introducing DA utility in conjunction with parameter uncertainty on the optimal asset allocation. We mainly focus on whether parameter uncertainty in a DA framework induces horizon effects (i.e., whether long-term allocation to the risky asset is different than short-term allocation).

[Figure 3 about here.]

Fig. 3 shows the optimal buy-and-hold portfolio allocations to the risky asset for a DA investor ($A = 0.44$ or $A = 0.30$) and an investor with power utility ($A = 1$; solid line) when returns are i.i.d. and parameter uncertainty about μ and σ is either considered (solid line) or ignored (dashed line). A DA investor who acknowledges parameter uncertainty will decrease her portfolio allocation to the risky asset with the investment horizon compared to the one with the power utility who will hold the same portfolio regardless of the horizon. This comes as the result of the evolution of cumulative returns' variance at difference horizons. Under parameter uncertainty, variance grows faster than linearly, which is the case when parameter uncertainty is ignored. Investors under parameter uncertainty consider equity most attractive when predictability is ignored, and this ignorance results in lower portfolio allocation to the risky asset. By Lemma 1 we see that the magnitude of the horizon effects depends on the available data incorporated into the model, in the following way: given σ, the variance of μ is inversely proportional to N (the sample size of risky asset return); subsequently, the larger the N, the lower the variance of μ and, equivalently, the smaller the uncertainty around its true value. A smaller available sample (i.e., a shorter investment horizon) would result in significantly lower allocation to the risky asset for an investor who considers uncertainty compared to one who ignores it, especially for longer horizons.

Incorporating DA drastically changes the portfolio's composition over different investment horizons. A DA investor ($A = 0.44$ or $A = 0.30$) will increase her investment proportion to the risky asset when allocating wealth for longer periods. The effect of DA appears to be more powerful at short horizons ($T < 10$), as a DA investor holds significantly less equity compared to one with power utility. A DA investor invests 20% to 50% of her wealth in the risky asset when her investment horizon is shorter than ten years (between 60% and 20% less equity compared to one with power utility), whereas an even more DA investor ($A = 0.30$) will hold no more than 10% to 40% equity for the same horizon. However, eventually, investors with DA utility will allocate similar to those with power utility as the investment horizon increases. A DA investor appears to be very conservative in the short run, whereas, when investing for long horizons, even a very DA investor ($A = 0.30$, i.e., losses in her utility function are weighed more than 3 times more than gains) is willing to accept the additional risk in anticipation of higher terminal wealth, because of the lower volatility as a result of the longer investment horizon.

Predictability is critical in the case of a buy-and-hold investor. Investors who take into account predictability will allocate significantly larger weights to equity for longer investment horizons, as volatility does not grow in proportion to asset returns. The latter results in lower long-term volatility, compared to the short-term, thus making equities appear more attractive to an investor with a long-term outlook. Fig. 4 displays optimal allocations to the risky asset for three levels of risk aversion (the ones most commonly used in relevant studies) and four levels of DA, among which is the value of $\frac{1}{\lambda}$, where λ is the loss aversion coefficient equal to $\lambda = 2.25$, as calculated by Tversky and Kahneman (1992). As expected, both risk aversion and DA affect the asset allocation to the risky asset as the more risk averse or disappointment averse an investor becomes, the lower the allocation in the risky asset will be.

[Figure 4 about here.]

The horizon effects we report for the buy-and-hold investor who uses the VAR to forecast equity returns can be traced to the evolution of return volatility. Long-term volatility is lower than in the case of i.i.d. returns due to the correlation between the predictor variable and the predicted equity return.[15]

[Figure 5 about here.]

As a result, the long-term volatility for a buy-and-hold investor who uses the VAR is much smaller than that for the investor who uses the i.i.d. return generator, growing slower than linearly. In particular, under i.i.d. returns, the 40-year total volatility equals $0.1625\sqrt{40} = 1.02$, while the standard deviation for the 40-year total return as predicted by the VAR equals 0.5091; that is, it is half as much (see Fig. 5). This result shows how the investment allocation in stocks can be affected (i.e., increase) by utilizing a variable believed to predict stock returns.

The intuition behind this effect is twofold. On the one hand, when the dividend yield decreases, the asset price will increase, in effect, disproportionately compared to the dividend yield. This signals that the current price is too high or equivalently that the expected return is too low (as the current price is too high). The too high current price mean-reverts, resulting in the negative association ($\rho < 0$; see Table 2) between the dividend yield and the future realized return, which, in turn, reduces the rate of increase of the variance, thus rendering equity more attractive at longer horizons. On the other hand, investors relying on a given strategy (in our case, the dividend yield) could develop an illusion of control if they grow overly confident in the strategy's ability to generate precise predictions of future returns. Overreliance is bound to boost investors' overconfidence levels, and lead them to assume higher

[15] More specifically, when we model returns as i.i.d., the two-period variance is equal to

$$var_{r_1,r_2} = var_{r_1} + var_{r_2} \Leftrightarrow \sigma_{r_1,r_2} = \sqrt{var_{r_1} + var_{r_2}}.$$

When returns are predictable, the covariance between equity returns and the predictor variable should be taken into consideration as well. The two-period variance is now equal to

$$var_{r_1,r_2} = var_{r_1} + var_{r_2} + 2cov(r_1, r_2).$$

Given that the covariance term in our VAR estimation is negative (see ρ, σ_{11}, and σ_{22} in the first column of Table 2), the following holds:

$$var_{r_1} + var_{r_2} + 2cov(r_1, r_2) < var_{r_1} + var_{r_2}.$$

risk in their investments by increasing their equity exposure (Odean, 1998; Gervais and Odean, 2001). Overconfidence is expected to be further encouraged by the fact that investors whose outlook involves long horizons and/or buy-and-hold strategies monitor their investments less frequently; neglect leads them to experience feelings of regret and/or disappointment equally less frequently, prompting them to view equity as less risky (because longer horizons experience fewer price fluctuations than do shorter ones) and thus tacitly encourage them to increase their exposure to risk (Benartzi and Thaler, 1995).

When parameter uncertainty is incorporated (right column of Fig. 4), a DA investor who accounts for predictability will allocate a smaller proportion of her investment to the risky asset compared to an investor who ignores parameter uncertainty. When parameter uncertainty is incorporated, equities do not look as attractive as when parameter uncertainty is ignored, because of the higher volatility of equity returns; the latter is due to uncertainty dampening the correlation between the predictor variable and the dependent variable (i.e., equity return), which, in turn, increases the volatility. Expressing uncertainty about the model's parameters is, in essence, equivalent to expressing uncertainty about the forecasting capacity of the predictor variable (i.e., the dividend price ratio). This uncertainty, in turn, can prompt investors to start viewing the VAR process as potentially misspecified, thus rendering them more ambiguity averse and leading them to reduce their exposure to equity investments (Maenhout, 2004). Under parameter uncertainty a DA investor will still hold larger weights for longer horizons compared to shorter ones, but they will be significantly lower than those allocated when parameter uncertainty is ignored.

4.2 Dynamic strategies

We now present the results for the case of a DA investor who follows a dynamic strategy and reallocates her available wealth at the beginning of each period between the risk-free and the risky asset. An investor who dynamically allocates wealth considers the investment opportunity set for the whole remaining investment period $T-t$ and assigns the optimal weight to the risky asset knowing that she will have the chance to revise her strategy by the end of the next period in case her expectations of the risky asset's return and volatility change. This is the difference between a dynamic and a static strategy in which investors follow a one-period forward-looking strategy.

4.2.1 Results with i.i.d. returns.

With i.i.d. returns, an investor who dynamically allocates and ignores parameter uncertainty uses the normality assumption and the i.i.d. asset return generator with parameters equal to the historical annual mean and volatility of the S&P 500 ($\mu = 0.1045$, $\sigma = 0.1635$; see Table 1). As expected she has the same investment opportunity set at every horizon, and the allocation to the risky asset does not change at different horizons (dashed line in Fig. 6).

[Figure 6 about here.]

4.2.2 Results with predictable returns.

The left column of Fig. 7 reports optimal portfolio allocations for four different levels of the DA coefficient A and three levels of the risk aversion coefficient γ at horizons $T - t$ between 1 and 40 years. The four levels of DA are the same as those used in the buy-and-hold case.

[Figure 7 about here.]

In these experiments, investors reallocate their available wealth at the end of each year, considering the optimal solutions from the solved subproblems at each horizon. For the same level of risk aversion, the more disappointment averse an investor is, the less she allocates to equities. The investment horizon effect of DA is visible by measuring the equity allocation at $T = 40$ and $t = 1$. The dynamic allocation to the risky asset drops as the investment horizon becomes shorter as a result of the lower per-period volatility for longer investment horizons shown in Fig. 5. A moderately DA investor will still invest a significant part of her portfolio to equity even at very short horizons (dashed line in Fig. 7), whereas a more DA investor will almost refrain from holding any units of the risky asset even when having a relatively low level of risk aversion.

When investors believe returns to be forecastable, they use the VAR to predict the next period's equity return and allocation drops for the investment horizon for all four different values of A. As the investment horizon $T-t$ shortens, a DA investor who follows a dynamic strategy allocates a smaller proportion of her wealth to the risky asset, whereas a DA and risk-averse investor will hold no units of the risky asset as $T-t$ approaches zero. Again, dynamically investing in the risky asset in the short run is not as attractive as in the long run given the higher volatility per period of the former. As a consequence, the more disappointment averse an investor is, the more likely she is to be affected by short-run volatility. This gives rise to horizon effects when investors try to hedge their portfolios at shorter horizons against a possible sharp move in the value of the independent variable (dividend yield).

4.2.3 Parameter uncertainty.

Let us assume an investor who uses the i.i.d. return generator and considers uncertainty in the model's parameters. In this case she will exhibit slightly different portfolio allocations compared to when the model's parameters are treated as known. Fig. 6 shows that both a DA investor and one who uses the power utility function will slightly increase her portfolio allocation to the risky asset with the investment horizon (solid line) to eventually hold a portfolio position very similar to an investor who ignores parameter uncertainty (dashed line). Investing for a longer horizon appears to be more risky than holding the risky asset in the short run as a result of the lower per-period volatility of the latter. As a result, an investor who dynamically invests with a shorter-term outlook will hold slightly more equity in their portfolio compared to an investor who invests for a longer horizon.

Turning to the case of predictability, the right column of Fig. 7 reports results that reflect optimal allocations to the risky asset for investors who rebalance their portfolios by predicting asset returns based on the dividend yield when parameter uncertainty is accounted for. These plots mainly reveal two facts: first, equity allocation is, in general, lower compared with the case of an investor who ignores parameter uncertainty, and, second, the impact of DA again appears to be more powerful at shorter horizons (up to 10 years), whereas for longer ones allocation lines become relatively flat. When we express uncertainty about the parameters of the VAR, we use the posterior predictive distribution in Lemma 2 in place of the VAR model with fixed parameters as stated in Equation (20). In this case, instead of simulating future return paths conditioning on fixed values for the model parameters (constant terms, matrix of AR coefficients, and variance-covariance matrix), we sample from their posterior distributions, each time obtaining a new set of parameters that is conditional on observed data only.

The results exhibit a pattern similar to the one in the left column of Fig. 7. The more disappointment averse and risk averse an investor grows, the lower the equity allocation will be at different investment horizons. Like in the case of a DA investor who follows a buy-and-hold strategy, the choice of the DA level mainly affects the dynamic allocation at longer horizons, whereas, as $T - t$ approaches zero, the allocation lines converge. The underlying cause for this behavior can be explained by the way the mean return and variance change over time. Investors' uncertainty about the predictive capacity of the dividend yield results in higher long-term per-period volatility, which explains investors' lower allocation to the risky asset compared to allocation in the left column of Fig. 7, where parameter uncertainty is ignored.

In other words, in some cases parameter uncertainty makes investors skeptical about whether investment opportunities actually change over time. Subsequently, investors doubt that higher or lower equity allocations will result in more optimal portfolios. In these cases, their allocations will be similar at different horizons compared with those cases in which they ignore parameter uncertainty.

5 Concluding Remarks

Disappointment aversion (DA) is a critical factor in portfolio choice, because it can drastically affect investors' portfolio compositions. Our experiments suggest that in the context of an utility maximization problem, a DA investor would allocate lower weights to equities compared to an investor who uses a standard CRRA power utility function.

DA introduces horizon effects for a buy-and-hold investor regardless of whether she employs either of the return generators and accounts for or ignores parameter uncertainty. We use the VAR to predict equity returns. Our examination of the evolution of the risky asset's return volatility throughout the investment horizon reveals that for the risky asset grows slower than linearly in the case in which the i.i.d. return generator is used. This seems to offer a plausible explanation for the observed horizon effects, whereas, in the case of a VAR with parameter uncertainty, additional uncertainty is expressed as increased volatility in risky asset's return, which, in turn, decreases the allocation to the risky asset.

Focusing on dynamic investing, we examine cases in which investors believe returns are i.i.d. or forecastable (through the dividend yield) and in which parameter uncertainty is ignored as well as incorporated in the asset allocation framework.

When predictability is considered, the distribution of the future returns generated by the VAR is significantly different from that of i.i.d. returns, because of the presence of a correlation between the dividend price ratio and the return of the risky asset. With i.i.d. returns, a horizon effects arise when parameter uncertainty is considered. Investors allocate smaller proportions to stocks for shorter horizons after accounting for the increased variance in equity returns.

Finally, the incorporation of parameter uncertainty in the DA framework with predictability changes equity allocations drastically. Overall, it is beneficial to be examined as a special case in a portfolio model, as frameworks that do not account for this may generate portfolios with too large equity allocations. When model parameters are taken as uncertain, a DA investor will still allocate larger weights to stocks at longer horizons. Nevertheless, the difference between a long-term and a short-term equity weight is smaller compared to the case in which parameter uncertainty is ignored, and as a result of the doubts investors cast on the predictive power of the dividend yield.

Our results should be of particular interest to policy makers, as they indicate that DA, conditional on its magnitude, tacitly fosters limited-to-no participation in equity investing. To the extent that DA is likely to affect individual investors more (given their lower sophistication levels, Barber et al., 2009), financial literacy programs could raise awareness of DA, while training people to assess their investments from a longer-term perspective, regardless of price movements in the short run (where the effects of DA are more likely to be felt). This, in turn, will help enhance the participation of retail investors in equity turnover (thus benefiting market liquidity), while ensuring that those that invest in equities are less likely to exit the market because of disappointment-related reasons. Our results are also relevant to finance practitioners, in particular brokers and financial advisors, who, by virtue of their profession, tend to engage with retail investors on a regular basis. For these practitioners, accounting for DA in their clients' risk profiling and overall day-to-day interactions would considerably help inform their professional practice, by permitting practitioners additional insight into their clients' trading decisions. Such insight could allow them to educate their clients about the role of DA in trading, thus helping them potentially improve their trading decisions. From an academic perspective and to the extent that disappointment stems from prior investment experience, our results also offer an alternative explanation of previously documented evidence (Seru et al., 2010; Strahilevitz et al., 2011) of the reluctance of investors to reenter the market if they have exited it previously at a loss.

Exploration of behavioral utility functions, particularly the DA theory, in the context of an asset allocation optimization problem is far from complete. In practice, investors' portfolios contain riskless and numerous asset classes of risky assets. Hence, a natural extension of the current framework seems to be building one able to deal with multi-asset portfolios, with additional asset classes, and incorporating consumption and trading costs. Moreover, given that prices of financial assets are determined by the forces of supply and demand, which stem from the trading and investment decisions of market participants, it would be of great interest to depart from partial equilibrium and study this interplay in a general equilibrium context. Doing so would allow us to investigate the interaction between investors with CRRA preferences and those using a DA utility function and, eventually, to reveal the role that DA plays in both defining asset prices and determining trading behaviors.

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APPENDICES

A Static Portfolio Allocation

In this appendix we revisit the static asset allocation problem, with the purpose of updating Ang et al. (2005) using the most recent data sample (1934-2016). We start by formulating the first-order condition (FOC, thereafter) for the single-period investing, which is given by:

Proposition 3 *The FOC for the maximization of $U(\mu)$*[16] *in Equation (1) is given by*

$$\mathbb{E}\left[\frac{dU(\mu)}{d\mu}X\,\boldsymbol{1}_{[W\leq\mu]}\right]+A\mathbb{E}\left[\frac{dU(\mu)}{d\mu}X\,\boldsymbol{1}_{[W>\mu]}\right]=0,\ \ \alpha\neq 0, \tag{23}$$

where $X=e^{y}-e^{r_f}$ is the return of equity over that of the riskless asset.

Proof. We calculate the FOC for the single-period case. Let

$$W=\alpha(e^{y}-e^{r})+e^{r}=\alpha X+e^{r}, \tag{24}$$

where α is the portfolio weight allocated to the risky asset, y is the risky asset's return, r is the risk-free interest rate known at the time of every investment decision and X is the equity premium. Considering an arbitrary utility function U, we extend it to define μ (i.e., the certainty equivalent of wealth) in the following way:

$$U(\mu)=\frac{1}{K}\left\{\mathbb{E}(U(W))\mathbf{1}_{W\leq\mu}+A\mathbb{E}(U(W))\mathbf{1}_{W>\mu}\right\}, \tag{25}$$

where $K=\mathbb{E}(\mathbf{1}_{W\leq\mu})+A\mathbb{E}(\mathbf{1}_{W>\mu})$. Now, maximizing over α, the Equation (26) derives:

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha}=\frac{1}{K}\left\{\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W\leq\mu}+A\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W>\mu})\right\}- \\ \frac{U(\mu)}{K}\left\{\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W\leq\mu})+\frac{d}{d\alpha}A\mathbb{E}(U(W)\mathbf{1}_{W>\mu})\right\}, \tag{26}$$

[16] For simplicity, in the derivations of this section we drop the subscript W from the certainty equivalent of wealth μ_W.

where the second part of the last equation follows after plugging Equation (25) into Equation (26) and simplifying the expression. We notice that

$$\mu = \alpha X + R \Leftrightarrow X = \frac{\mu - R}{\alpha}, \tag{27}$$

therefore the derivatives of the expected values can be expressed as follows,

$$\frac{d}{d\alpha}\mathbb{E}(U(W))\mathbf{1}_{W>\mu} = \frac{d}{d\alpha}\left(\int_{\frac{\mu-R}{\alpha}}^{\xi} f(X)U(\alpha X + R)dX + \int_{\xi}^{\infty} f(X)U(\alpha X + R)dX\right), \tag{28}$$

which by the Leibniz rule can be written as

$$\begin{gathered}\int_{\frac{\mu-R}{\alpha}}^{\infty} f(X)\frac{dU(\alpha X + R)}{d\alpha}XdX - f\left(\frac{\mu-R}{\alpha}\right)U(\mu)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right) = \\ \mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W>\mu}\right) - f\left(\frac{\mu-R}{\alpha}\right)U(\mu)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right),\end{gathered} \tag{29}$$

where $f(X)$ is the normal probability density function. Similarly, the Equations (30), (31) and (32) are calculated,

$$\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W\leq\mu}) = \mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W\leq\mu}\right) + f\left(\frac{\mu-R}{\alpha}\right)U(\mu)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right), \tag{30}$$

$$\frac{d}{d\alpha}\mathbb{E}(\mathbf{1}_{W\leq\mu}) = \frac{d}{d\alpha}\int_{-\infty}^{\frac{\mu-R}{\alpha}} f(X)dX = \frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)f\left(\frac{\mu-R}{\alpha}\right) \tag{31}$$

and

$$\frac{d}{d\alpha}\mathbb{E}(\mathbf{1}_{W>\mu}) = \frac{d}{d\alpha}\int_{\frac{\mu-R}{\alpha}}^{\xi} f(X)dX = -\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)f\left(\frac{\mu-R}{\alpha}\right). \tag{32}$$

Substituting Equations (30) to (31) into (29), we take:

$$\begin{aligned}\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = {}& \frac{1}{K}\left[\mathbb{E}\left(\frac{XdU(W)}{dW}\mathbf{1}_{W\leq\mu}\right) + U(\mu)f\left(\frac{\mu-R}{\alpha}\right)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right) + A\mathbb{E}\left(\frac{XdU(W)}{dW}\mathbf{1}_{W>\mu}\right)\right.\\ & \left. - AU(\mu)f\left(\frac{\mu-R}{\alpha}\right)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)\right] - \frac{U(\mu)}{K}\left[\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)f\left(\frac{\mu-R}{\alpha}\right)\right.\\ & \left. - Af\left(\frac{\mu-R}{\alpha}\right)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)\right] = \frac{1}{K}\left[\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W\leq\mu}\right) + A\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W>\mu}\right)\right].\end{aligned} \tag{33}$$

The FOC is now given by

$$\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W\leq\mu}\right)+A\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W>\mu}\right)=0,\quad \alpha\neq 0. \tag{34}$$

■

Equations (1) and (23) have to be solved simultaneously, since $\mu=\mu(A,\alpha)$ is itself a function of α and A. A way to proceed with the solution of the system of equations is to discretize it using a suitable quadrature method. For more details on this process, see Appendix B.

B Solution to DA Portfolio Choice Problem

This appendix describes the discretization procedure of the DA asset allocation problem and the solution to the system of simultaneous continuous Equations (1) and (23).

Analytically, to solve the system of Equations (10) and (13) (the discrete equivalents to Equations (1) and (23)), the Gaussian Quadrature method is used. Since a lognormal distribution for the returns is assumed, the logarithmic returns are normally distributed. Under this assumption the numerical scheme of Gauss-Hermite is used to convert integrals of exponential expressions into the form of

$$\int_{-\infty}^{\infty} exp(-x^2) f(x) dx \approx \sum_{i=1}^{N} f(x_i) w_i, \tag{35}$$

where $\{x_i\}_{i=1}^N$ are the discrete points the integral in Equation (35) is calculated at and $\{w_i\}_{i=1}^N$ are the corresponding weights. The points x_i, known as abscissae, are the roots of the Hermite polynomials, while the weights w_i are derived after a suitable transformation. Based on the discretization procedure, we write Equation (10) as

$$\mu_t = W_t \left(\prod_{i=t+1}^{T-1} \mu_i^* \right) \left[\frac{\sum_{s=1}^{M} p_s W_{s,t+1}^{1-\gamma} + A \sum_{s=M+1}^{N} p_s W_{s,t_1+1}^{1-\gamma}}{P(W_{t+1} \leq \mu_t) + AP(W_{t+1} > \mu_t)} \right]^{1/(1-\gamma)}, \tag{36}$$

and Equation (34), the FOCs of the problem, as

$$\sum_{s=1}^{M} p_s W_{s,t+1}^{-\gamma} X_{s,t+1} + A \sum_{s=M+1}^{N} p_s W_{s,t+1}^{-\gamma} X_{s,t+1} = 0, \tag{37}$$

where $X_{t+1} = e^{y,t+1} - e^{r,t}$ is the excess return of the risky asset over the period $t, t+1$, and s takes one of the values 1 to N, where N is the number of risky asset's discrete states. We notice that the outcomes are split to the two sums with respect to their relationship to the certainty equivalent. Given that the discrete return states are ordered from smallest to largest, the first sum takes on all the discrete outcomes that lie below μ_t while the second one takes those above the certainty equivalent and scales them down via the DA coefficient A.

The solution to the Equations (36) and (37) yields the portfolio weight α that maximizes the DA utility and the value of the certainty equivalent for each corresponding period. This solution is non-trivial, and thus we need to follow an algorithmic procedure similar to that in Ang et al. (2005). Considering the N states for the excess return at time $t+1$, $y_{t+1} - r_t$, we construct $N-1$ ordered intervals for the portfolio return as follows:

$$\begin{aligned} &\left[(\alpha X_1 + e^r) \prod_{j=t+1}^{T-1} \mu^*_{W_j}, (\alpha X_2 + e^r) \prod_{j=t+1}^{T-1} \mu^*_{W_j}\right), \ldots, \\ &\left[(\alpha X_{N-1} + e^r) \prod_{j=t+1}^{T-1} \mu^*_{W_j}, (\alpha X_N + e^r) \prod_{j=t+1}^{T-1} \mu^*_{W_j}\right). \end{aligned} \tag{38}$$

Assuming that the certainty equivalent $\mu \equiv \mu_W$ lies in the interval defined by the return states i and $i+1$, i.e., $[(\alpha X_i + e^r) \prod_{j=t+1}^{T-1} \mu^*_{W_j}, (\alpha X_{i+1} + e^r) \prod_{j=t+1}^{T-1} \mu^*_{W_j})$ where $1 < i \leq N$, $\prod_{j=t+1}^{T-1} \mu^*_{W_j}$ is the indirect utility of wealth and α satisfies the FOC, (37),

$$\sum_{s: W_s \prod_{j=t+1}^{T-1} \mu^*_{W_j} \leq \alpha^* X_i + e^r} p_s W_s^{-\gamma} X_s + A \sum_{s: W_s \prod_{j=t+1}^{T-1} \mu^*_{W_j} > \alpha^* X_{i+1} + e^r} p_s W_s^{-\gamma} X_s = 0, \tag{39}$$

where α^* is now the optimal value for the portfolio weight. As Equation (36) indicates, the probabilities for the outcomes above the certainty equivalent should be downweighted. Therefore, the corresponding probabilities are multiplied by the DA coefficient A and then divided by the sum of all probabilities related to the possible return states so as to add up to one. The new probability distribution is defined as

$$\pi_s = \frac{p_s 1_{\{s \leq i\}} + A p_s 1_{\{s \geq i+1\}}}{\sum_{s=1}^{i} p_s + A \sum_{s=i+1}^{N} p_s}, \; 1 < s \leq N. \tag{40}$$

Given initial guess i for the state of the certainty equivalent we solve for α^* and μ^*_W which is now stated as

$$\mu_{W_i} = \left(\pi_s \sum_{s=1}^{N} (W_s)^{1-\gamma} \prod_{j=t+1}^{T-1} \mu^*_{W_j}\right)^{\frac{1}{1-\gamma}}, \tag{41}$$

where the second term of Equation (36) is absorbed by the changed probability distribution in Equation (40). In case $\mu^*_{W_i}$ lies within the interval defined by our initial guess

$$\mu_{W_i} \in \left[(\alpha_i X_i + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}, (\alpha_i X_{i+1} + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}\right), \tag{42}$$

$\mu^*_W = \mu_{W_i}$, $\alpha^* = \alpha_i$ and i is the optimal state for the problem. If the condition in Equation (42) is not satisfied we perform a binary search given ordered return intervals, until α falls within the right interval.

C Different Proofs

This appendix contains the proofs of propositions and the theorem stated in the main body of this paper.

C.1 Proof of Proposition 1

We define the DA utility function for the dynamic asset allocation problem as follows:

At time $t = T-1$ we have

$$U(\mu_{T-1}) = \frac{1}{K_{T-1}}\Big[\mathbb{E}_{T-1}(U(W_{T-1}R_T(\alpha_{T-1}))\mathbf{1}_{\{W_{T-1}R_T(\alpha_{T-1})\leq\mu_{T-1}\}}) \\ + A\mathbb{E}_{T-1}(U(W_{T-1}R_T(\alpha_{T-1}))\mathbf{1}_{\{W_{T-1}R_T(\alpha_{T-1})>\mu_{T-1}\}})\Big]. \quad (43)$$

Continuing recursively, at time $t = T-2$ the DA utility is defined as

$$U(\mu_{T-2}) = \frac{1}{K_{T-2}}\Big[\mathbb{E}_{T-2}(U(W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1}))\mathbf{1}_{\{W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1})\leq\mu_{T-2}\}}) \\ + A\mathbb{E}_{T-2}(U(W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1}))\mathbf{1}_{\{W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1})\leq\mu_{T-2}\}})\Big]. \quad (44)$$

Eventually, at time t, we will have

$$U(\mu_t) = \frac{1}{K_t}\Big[\mathbb{E}_t(U(W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}))\mathbf{1}_{\{W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}\leq\mu_t\}}) \\ + A\mathbb{E}_t(U(W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}))\mathbf{1}_{\{W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}>\mu_t\}})\Big], \quad (45)$$

where $Q^*_{t+1,T} = R_{t+2}(\alpha^*_{t+1})\cdots R_T(\alpha^*_{T-1})$ is the optimal aggregate return between $t+1$ and T that maximizes the correspodning utility of wealth $U(W)$.

We next calculate the optimization condition for the multiperiod dynamic problem. In this case, we have T periods and our investor maximizes the utility of wealth

$$\max_{\alpha_0,\alpha_1,\cdots,\alpha_{T-1}} \mathbb{E}_0[U(W_T)], \quad (46)$$

where the wealth is given by $W_t = R_t(\alpha_{t-1})W_{t-1}$, where $R_t(\alpha_{t-1}) = \alpha_{t-1}(e^{y_t} - e^{r_{t-1}}) + e^{r_{t-1}}$. Considering we are at time $T-1$, we follow Equation (33) and by adding time subscripts we end up with the following expression:

$$
\begin{aligned}
\frac{dU(\mu_{T-1})}{d\mu_{T-1}}\frac{d\mu_{T-1}}{d\alpha_{T-1}} &= \frac{1}{K_{T-1}}\Bigg[\mathbb{E}_{T-1}\left(X_T\frac{dU(W_T)}{dW}\mathbf{1}_{W_T\le\mu_{T-1}}\right) \\
&\quad + A\mathbb{E}_{T-1}\left(X_T\frac{dU(W_T)}{dW}\mathbf{1}_{W_T>\mu_{T-1}}\right)\Bigg] \\
&= \frac{1}{K_{T-1}}\Bigg[\mathbb{E}_{T-1}\left(W_{T-1}\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T\le\mu_{T-1}}\right) \\
&\quad + A\mathbb{E}_{T-1}\left(W_{T-1}\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T>\mu_{T-1}}\right)\Bigg].
\end{aligned}
\tag{47}
$$

But at time $T-1$, the terms in W_{T-1} become known and eventually W_{T-1} can be taken outside the expectation term leading to the following FOC

$$
\mathbb{E}_{T-1}\left(\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T\le\mu_{T-1}}\right)+A\mathbb{E}_{T-1}\left(\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T>\mu_{T-1}}\right)=0.
\tag{48}
$$

Moving backwards to time t, the wealth W_T can be expressed as

$$
W_T = Q^*_{t+1,T}R_{t+1}(\alpha_t)W_t \tag{49}
$$

and Equation (47) can be rewritten in the following way

$$
\begin{aligned}
\frac{dU(\mu_t)}{d\mu_t}\frac{d\mu_t}{d\alpha_t} = \frac{1}{K_t}\Bigg[&\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\le\mu_t}\right) \\
&+ A\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\right)\Bigg],
\end{aligned}
\tag{50}
$$

and the FOC is as follows

$$
\begin{aligned}
&\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\le\mu_t}\right) \\
&\quad + A\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\right)=0.
\end{aligned}
\tag{51}
$$

C.2 Proof of Proposition 2

We formulate the FOCs for the optimization problem by performing the substitution of the value $R_{i+1}(\alpha_i^*), i = t, t+1\cdots, T-1$ with the certainty equivalent for the same period, μ_i^*. Using this approach, we keep the dimension of the state space constant with time, which allows us to solve the problem computationally in reasonable time. Recalling the implicit definition for the certainty equivalent and the expression for $Q^*_{t+1,T}$ we have that

$$
\begin{aligned}
U(\mu_t) &= \frac{1}{K_t}\Big[\mathbb{E}_t(U(W_t)\mathbf{1}_{W_T\le\mu_t}) + A\mathbb{E}_t(U(W_t)\mathbf{1}_{W_T>\mu_t})\Big] \\
&= \frac{1}{K_t}\Big[\mathbb{E}_t(U(Q^*_{t+1,T}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{W_T\le\mu_t}) + A\mathbb{E}_t(U(Q^*_{t+1,T}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{W_T\le\mu_t})\Big] \\
&= \frac{1}{K_t}\Big[\mathbb{E}_t\Big(U(\mu^*_{T-1}\ldots\mu^*_{t+1}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{\{R_{t+1}(\alpha_t)\le\frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}\}}\Big) \\
&\quad + A\mathbb{E}_t\Big(U(\mu^*_{T-1}\ldots\mu^*_{t+1}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{\{R_{t+1}(\alpha_t)>\frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}\}}\Big)\Big]. \qquad (52)
\end{aligned}
$$

Equation (52) can be also written as

$$
\begin{aligned}
U(\mu_t) = \frac{1}{K_t}\Big[&\mathbb{E}_t\Big(U(R_{t+1}(\alpha_t)W_t\prod_{i=t+1}^{T-1}\mu^*_i)\mathbf{1}_{\{R_{t+1}(\alpha_t)\le\xi_t\}}\Big) \\
&+ A\mathbb{E}_t\Big(U(R_{t+1}(\alpha_t)W_t\prod_{i=t+1}^{T-1}\mu^*_i)\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}}\Big)\Big], \qquad (53)
\end{aligned}
$$

by accumulating the certainty equivalent return terms μ^* under the product notation and setting $\xi_t = \frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}$. For the calculation of the FOC of $U(\mu_t)$ we will perform an algebraic transformation so as to simplify the calculus. In Equation (52), the product $\mu^*_{T-1}\ldots\mu^*_{t+1}W_t$ is known at time t and thus, for simplicity, it can be written outside the expectation terms as a function of the utility of wealth $U(\cdot)$. This transformation is as follows:

$$
\begin{aligned}
U(\mu_t) = \frac{f(U(\prod_{i=t+1}^{T-1}\mu^*_iW_t))}{K_t}\Big[&\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{\{R_{t+1}(\alpha_t)\le\xi_t\}}) \\
&+ A\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}})\Big], \qquad (54)
\end{aligned}
$$

where $\xi_t = \frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}$ and $K_t = \mathbb{E}_t(\mathbf{1}_{W_T\le\mu_t}) + A\mathbb{E}_t(\mathbf{1}_{W_T>\mu_t})$. Hence, we obtain

$$
\begin{aligned}
\frac{1}{f(U(\mu^*_{T-1}\ldots\mu^*_{t+1}W_t))}\frac{dU(\mu_t)}{d\mu_t}\frac{d\mu_t}{d\alpha_t} &= -\frac{1}{K_t^2}\frac{dK_t}{d\alpha_t}K_tU(\mu_t) + \frac{1}{K_t}\Big[\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)\le\xi_t} \\
&\quad + A\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\Big] \\
&= \frac{1}{K_t}\Big[A\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t} + \frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)\le\xi_t}\Big] \\
&\quad - \frac{U(\mu_t)}{K_t}\Big[A\frac{d}{d\alpha_t}\mathbb{E}_t(\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}) + \frac{d}{d\alpha_t}\mathbb{E}_t(\mathbf{1}_{R_{t+1}(\alpha_t)\le\xi_t})\Big]. \qquad (55)
\end{aligned}
$$

But we know that

$$
R_{t+1}(\alpha_t) > \xi_t \Leftrightarrow \alpha_tX_{t+1} + e^{r_t} > \xi_t \Leftrightarrow X_{t+1} > \frac{\xi_t - e^{r_t}}{\alpha_t} \equiv \xi,\ \ \alpha > 0, \qquad (56)
$$

therefore, we can express the derivatives as in the proof of Proposition 1. More specifically, the first term of Equation (55) is

$$\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t} = \frac{d}{d\alpha_t}\int_{\xi}^{\infty} U(R_{t+1}(\alpha_t))F(X_{t+1})dX_{t+1}$$
$$= \frac{d}{d\alpha_t}\int_{\xi}^{\infty} U(\alpha_t X_{t+1} + e^{r_t})F(X_{t+1})dX_{t+1}$$
$$= \frac{d}{d\alpha_t}\int_{\xi}^{\xi'} U(\alpha_t X_{t+1}+e^{r_t})F(X_{t+1})dX_{t+1} + \frac{d}{d\alpha_t}\int_{\xi'}^{\infty} U(\alpha_t X_{t+1}+e^{r_t})F(X_{t+1})dX_{t+1}$$
$$= \int_{\xi}^{\xi'} \frac{d}{d\alpha_t}U(\alpha_t X_{t+1} + e^{r_t})X_{t+1}F(X_{t+1})dX_{t+1} - F(\xi)U\left(\alpha_t \frac{\xi_t - e^{rt}}{\alpha_t} + e^{r_t}\right)\frac{d\xi}{d\alpha_t}$$
$$+ \int_{\xi'}^{\infty} \frac{d}{d\alpha_t}U(\alpha_t X_{t+1} + e^{r_t})X_{t+1}F(X_{t+1})dX_{t+1}$$
$$= \int_{\xi}^{\infty} \frac{d}{d\alpha_t}U(\alpha_t X_{t+1} + e^{r_t})X_{t+1}F(X_{t+1})dX_{t+1} - F(\xi)U(\xi_1)\frac{d\xi}{d\alpha_t}$$
$$= \mathbb{E}_t\left(X_{t+1}\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right) - F(\xi)U(\xi_t)\frac{d\xi}{d\alpha_t}. \tag{57}$$

Expressing the remaining terms of Equation (55) in the same way we obtain the following result:

$$\frac{1}{f(U(\mu^*_{T-1}\cdots\mu^*_{t+1}W_t))}\frac{dU(\mu_t)}{d\mu_t}\frac{d\mu_t}{d\alpha_t}$$
$$= \frac{1}{K_t}\left[\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)\le\xi_t} + A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right)\right)\right],$$

which considering the FOC of the expression above yields

$$\frac{dU(\mu_t)}{d\alpha_t} = 0 \Leftrightarrow$$
$$\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)\le\xi_t}\right) + A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right) = 0, \tag{58}$$

with $t = T-1,\ldots,0$. Comparing Equations (58) and (51), one can see the advantage of using the former over the latter as it involves only one uncertain variable, the return of the portfolio between t and $t+1$.

C.3 Proof of Theorem 1

We prove that for the critical level of disappointment aversion, A^*, any A below this value induces non-participation ($\alpha < 0$) while any A larger than A^* leads to positive portfolio allocation. Let $\mu = \mu_W(A, \alpha)$, with

- $\mu(A,:) \in \mathbf{C}^1, \forall A \in [0,1]$,
- $\frac{d\mu(A,0)}{d\alpha} = \xi(A) \leq 0, \forall A \in [0,1]$,
- $\mathbb{E}(X) > 0$ and $\mathbb{E}(X\mathbf{1}_{W\geq\xi(A)}) > 0$, where $X = e^y - e^r$ is the excess return of the equity over the bond.

Then, setting

$$A^* = \frac{\mathbb{E}(X\mathbf{1}_{W\geq\xi(A)})}{\mathbb{E}(X\mathbf{1}_{W<\xi(A)})}, \tag{59}$$

we have the following:

- For every $A \leq A^*$, $\quad \alpha^* = 0$;
- For every $A > A^*$, $\quad \alpha^* > 0$,

where α^* is the portfolio weight which maximizes $\mu(A,\alpha)$ for a given level of A.

Proof. We have that

$$W = \alpha(e^y - e^r) + e^r \stackrel{\Delta}{=} \alpha X + R, \tag{60}$$

which as $\alpha \to 0$ tends to R. The expected value of Equation (60) equals $\mathbb{E}(W) = \alpha\mathbb{E}(X) + r$. From the definition of the DA utility we have

$$\lim_{\alpha\to 0} U(\mu) = \lim_{\alpha\to 0} \frac{\mathbb{E}(U(W))\mathbf{1}_{W\leq\mu} + A\mathbb{E}(U(W))\mathbf{1}_{W>\mu}}{\mathbb{P}(W \leq \mu) + A\mathbb{P}(W > \mu)}, \tag{61}$$

which given that both the utility function and the certainty equivalent are $\mathbb{C}^1$-functions can be written as

$$U(\mu(A,0)) = \frac{\mathbb{E}(U(r)\mathbf{1}_{r\leq\mu(A,:)} + A\mathbb{E}(U(R))\mathbf{1}_{r>\mu(A,:)}}{\mathbb{P}(r \leq \mu(A,:)) + A\mathbb{P}(r > \mu(A,:))} \implies \mu(A,0) = r. \tag{62}$$

The last equality follows from the fact that the certainty equivalent U is a $1-1$ function. We now examine the behaviour of the function μ around zero. Thus, we consider two cases, one where α approaches zero from negative values and one where it approaches zero from positive ones.

- $a < 0$; from the FOCs for the optimization problem we have

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{1}{K}\left\{\mathbb{E}\left(X\frac{dU}{dW}\mathbf{1}_{X\geq(\mu-r)/\alpha}\right) + A\mathbb{E}\left(X\frac{dU}{dW}\mathbf{1}_{X<(\mu-r)/\alpha}\right)\right\}, \tag{63}$$

where $K = \mathbb{P}(W \leq \mu) + A\mathbb{P}(W > \mu) = \mathbb{P}(X \geq \frac{\mu-r}{\alpha}) + A\mathbb{P}(X < \frac{\mu-r}{\alpha})$.

Therefore,

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{1}{K}\left\{\int_{(\mu-r)/\alpha}^{+\infty} X\frac{dU(\alpha X + r)}{dW}f(X)dX + A\int_{-\infty}^{(\mu-r)/\alpha} X\frac{dU(\alpha X + r)}{dW}f(X)dX\right\}. \tag{64}$$

We have that

$$\frac{d\mu(A,0)}{d\alpha} = \xi(A) \equiv \xi \tag{65}$$

with

$$\lim_{\alpha\to 0} \frac{\mu(A,\alpha)-\mu(A,0)}{\alpha} = \xi \Leftrightarrow \lim_{\alpha\to 0} \frac{\mu(A,\alpha)-r}{\alpha} = \xi. \tag{66}$$

We now define

$$B(A,\alpha) = \begin{cases} \frac{\mu(A,\alpha)-r}{\alpha} & \text{if } \alpha <> 0 \\ \xi & \text{if } \alpha = 0 \end{cases} \tag{67}$$

where B is a continuous function. Therefore, Equation (63) can be rewritten as

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{\int_{B(A,\alpha)}^{+\infty} X \frac{dU(\alpha X + r)}{dW} f(X)dX + A\int_{-\infty}^{B(A,\alpha)} X \frac{dU(\alpha X+r)}{dW} f(X)dX}{\int_{B(A,\alpha)}^{+\infty} f(X)dX + A\int_{-\infty}^{B(A,\alpha)} f(X)dX}, \alpha \leq 0. \tag{68}$$

Subsequently we have,

$$\begin{aligned} \frac{du(\mu(A,0))}{d\mu} \lim_{\alpha\to 0^-} \frac{d\mu(A,\alpha)}{d\alpha} &= \frac{\int_{\xi}^{+\infty} X \frac{dU(r)}{dW} f(X)dX + A\int_{-\infty}^{\xi} X \frac{dU(r)}{dW} f(X)dX}{\int_{\xi}^{+\infty} f(X)dX + A\int_{-\infty}^{\xi} f(X)dX} \Leftrightarrow \\ \lim_{\alpha\to 0^-} \frac{d\mu(A,\alpha)}{d\alpha} &= \frac{\mathbb{E}(X\mathbf{1}_{W\geq\xi}) + A\mathbb{E}(X\mathbf{1}_{W<\xi})}{\mathbb{P}(X\geq\xi) + A\mathbb{P}(X<\xi)}. \end{aligned} \tag{69}$$

Since $\xi(A) \leq 0$ and $A \in [0,1]$ we have that

$$X\mathbf{1}_{W\geq\xi} + AX1_{W<\xi} \geq X \Rightarrow \lim_{\alpha\to 0^-} \frac{d\mu(A,\alpha)}{d\alpha} > 0. \tag{70}$$

Thus, there is $\epsilon > 0$ such that $\frac{d\mu(A,\alpha)}{d\alpha} > 0$ for every $\alpha \in (-\epsilon, 0) \Rightarrow \mu(A,\alpha)$, is strictly increasing with respect to α in $(-\epsilon, 0)$.

- $\alpha > 0$; in the case where zero is approached from the right we have

$$\begin{aligned} \frac{du(\mu(A,0))}{d\mu} \lim_{\alpha\to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} &= \frac{\int_{-\infty}^{\xi} X \frac{dU(r)}{dW} f(X)dX + A\int_{\xi}^{+\infty} X \frac{dU(r)}{dW} f(X)dX}{\int_{-\infty}^{\xi} f(X)dX + A\int_{\xi}^{+\infty} f(X)dX} \Leftrightarrow \\ \lim_{\alpha\to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} &= \frac{\mathbb{E}(X\mathbf{1}_{W\leq\xi}) + A\mathbb{E}(X\mathbf{1}_{W>\xi})}{\mathbb{P}(X\leq\xi) + A\mathbb{P}(X>\xi)}, \end{aligned} \tag{71}$$

which leads to

$$\lim_{\alpha\to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} = \frac{\mathbb{E}(X\mathbf{1}_{W\leq\xi}) + A^*\mathbb{E}(X\mathbf{1}_{W>\xi})}{K}, \tag{72}$$

since $A < A^*$ and the expected value of the return premium, X is positive. Now, given that

$$A^* = \frac{\mathbb{E}(X\mathbf{1}_{W\leq\xi})}{\mathbb{E}(X\mathbf{1}_{W>\xi})}, \tag{73}$$

the $\lim_{\alpha\to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} < 0$. Thus, there is $\delta > 0$ such that $\lim_{\alpha\to 0^+} \frac{d\mu(A,\alpha)]}{d\alpha} < 0$ for every $\alpha \in (0,\delta)$, $\mu(A,\alpha)$ is a strictly decreasing function with a local maximum at $\alpha = 0$ where $\mu(A,0) = r$. Therefore,

$$\mu(A,\alpha) \leq \mu(A,0), \forall\, \alpha \in (-\epsilon,\delta) \Rightarrow$$
$$U(\mu(A,\alpha)) \leq U(\mu(A,0)) = U(r) \Rightarrow \max_{\alpha} U(\mu(A,\alpha)) = \max_{\alpha} U(\mu(A,0)) = U(r). \tag{74}$$

We should notice that if $A < A^*$, the weight α is positive. Indeed we obtain

$$\lim_{\alpha\to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} > \frac{A^*\mathbb{E}(X\mathbf{1}_{W>\xi}) + A^*\mathbb{E}(X\mathbf{1}_{W\leq\xi})}{A\mathbb{E}(X\mathbf{1}_{W>\xi}) + A^*\mathbb{E}(X\mathbf{1}_{W\leq\xi})} = 0, \tag{75}$$

which implies that there exists

$$\delta > 0 : \frac{d\mu(A,\alpha)}{d\alpha} > 0, \forall\alpha \in (0,\delta) \Rightarrow \mu(A,\alpha) \uparrow (0,\delta). \tag{76}$$

Thus,

$$\begin{aligned} \mu(A,\alpha) \uparrow (-\epsilon,\delta) &\Rightarrow \exists\, \xi_0,\ \mu(A,\alpha) \uparrow [-\xi,\xi] \\ \max_{\alpha} \mu(A,\alpha) =&\mu(A,\ \xi),\ \xi > 0 \Rightarrow \alpha^* = \xi. \end{aligned} \tag{77}$$

■

D Bayesian Portfolio Analysis

This appendix includes the proofs to lemmas 1 and 2 regarding the posterior distribution with i.i.d. and predictable returns respectively.

D.1 Proof of Lemma 1 (i.i.d. returns)

We present the results for the Bayesian portfolio with investment in a risk-free and a risky asset.[17] The investor models her excess returns based on the following equation

$$r_t = \mu + \epsilon_t, \tag{78}$$

where r_t is the continuously compounded excess return at time t and $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. disturbance terms.

In line with most of the relevant literature, to estimate the posterior distribution $p(\mu, \sigma^2|Y)$, we consider an uninformative prior of the form

$$p(\mu, \sigma) d\mu d\sigma \propto \frac{1}{\sigma} d\mu d\sigma. \tag{79}$$

The joint posterior of μ and σ is

$$p(\mu, \sigma|Y) \propto p(\mu, \sigma) \times L(\mu, \sigma|Y), \tag{80}$$

where $L(.)$ is the likelihood function. The joint posterior density for μ and σ follows a normal distribution and is also equal to

$$\begin{aligned} p(\mu, \sigma|Y) &\propto \frac{1}{\sigma} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y_i - \mu)^2}{2\sigma^2}\right\} \\ &\propto \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right\} \\ &= \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2} n\left(\mu^2 + \frac{\sum_i y_i^2}{n} - \frac{2\mu \sum_i y_i}{n}\right)\right\}. \end{aligned} \tag{81}$$

[17] The derivations for both cases (i.i.d. returns and predictable returns) follow closely the models of (Zellner, 1996; Tiao and Zellner, 1964) but they are reported in a more analytical way here as especially in the case of i.i.d. returns a number of steps is omitted in the original papers.

Completing the square, Equation (81) can be written as

$$
\begin{aligned}
&= \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2} n\left(\mu^2 - \frac{2\mu\sum_i y_i}{n} + \frac{\sum_i y_i^2}{n} + \left(\frac{2\sum_i y_i}{2n}\right)^2 - \left(\frac{2\sum_i y_i}{2n}\right)^2\right)\right\} \\
&= \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2} n\left(\left(\mu - \frac{\sum_i y_i}{n}\right)^2 + \frac{\sum_i y_i^2}{n} - 2\left(\frac{\sum_i y_i}{n}\right)^2 + \left(\frac{\sum_i y_i}{n}\right)^2\right)\right\} \\
&= \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}\left(n\left(\mu - \frac{\sum_i y_i}{n}\right)^2 + \sum_i y_i^2 - 2\frac{\sum_i y_i}{n}\sum_i y_i + n\left(\frac{\sum_i y_i}{n}\right)^2\right)\right\}.
\end{aligned}
\tag{82}
$$

Performing the substitution $\overline{\mu} = \frac{\sum_i y_i}{n}$, Equation (82) can be rewritten as

$$
\begin{aligned}
p(\mu, \sigma|Y) &\propto \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}\left(n(\mu - \overline{\mu})^2 + \sum_i y_i^2 - 2\overline{\mu}\sum_i y_i + n\overline{\mu}^2\right)\right\} \\
&= \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}(n(\mu - \overline{\mu})^2 + \sum_i (y_i - \overline{\mu}^2))\right\}.
\end{aligned}
\tag{83}
$$

Dividing $\sum_i (y_i - \overline{\mu}^2)$ by $n-1$ in Equation (83) yields the unbiased variance estimator s^2 which gives the following:

$$
p(\mu, \sigma|Y) \propto \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}(n(\mu - \overline{\mu})^2 + (n-1)s^2)\right\}, \tag{84}
$$

where $(n-1)s^2 = \sum_i (y_i - \overline{\mu})^2$. From Equation (84), we see that the conditional mean and variance for the posterior mean are $E(\mu|\sigma, Y) = \overline{\mu}$ and $var(\mu|\sigma, Y) = \sigma^2/n$, respectively. To sample from the posterior for the mean $p(\mu|Y, \sigma)$ conditional on σ and the sample data Y we use the normal $\mathcal{N}(\overline{\mu}, \sigma^2 + \sigma^2/n)$. As this expression is conditional on σ we calculate the marginal posterior distribution for the standard deviation and then using this result we draw from the normal for the mean. We next marginalize out μ to derive the marginal posterior density for σ by expressing Equation (83) as

$$
\begin{aligned}
p(\sigma|Y) &= \int_{-\infty}^{\infty} p(\mu, \sigma|Y) d\mu \\
&= \int_{-\infty}^{\infty} \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}(n(\mu - \overline{\mu})^2 + (n-1)s^2)\right\} d\mu \\
&= \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \int_{-\infty}^{\infty} \exp\left\{ - \underbrace{\frac{1}{2\sigma^2} n(\mu - \overline{\mu})^2}_{\phi^2/2} \right\} d\mu.
\end{aligned}
\tag{85}
$$

Setting the absolute value of the exponent inside the integral equal to $\phi^2/2$, we rewrite Equation (85) as

$$
p(\sigma|Y) \propto \sigma^{-(n+1)} \exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{\phi^2}{2}\right\} \sigma d\phi, \tag{86}
$$

where the integration is now taking place with respect to ϕ as a result of the substitution, we performed. This follows from setting $\phi = \sigma^{-1}\sqrt{n}(\mu - \overline{\mu})$ which by differentiating both sides leads to $d\phi = \sigma^{-1}\sqrt{n}d\mu \propto \sigma^{-1}d\mu$ since everything else is constant. Next, we rewrite Equation (86) as follows

$$\begin{aligned} p(\sigma|Y) &= \sigma^{-(n+1)}\sigma \exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\phi^2\right\}d\phi \\ &= \sqrt{2\pi}\sigma^{-n}\exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \\ &\propto \sigma^{-n}\exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\}. \end{aligned} \tag{87}$$

In the second equality we substitute the integral with its solution as it represents a Gaussian integral which has a known general solution given by $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$. Setting $n-1 = N$ we have $n = N+1$ and Equation (87) can be written as

$$p(\sigma|Y) \propto \sigma^{-(N+1)}\exp\left\{-\frac{Ns^2}{2\sigma^2}\right\}. \tag{88}$$

As Zellner (1996) observes, the PDF in Equation (88), is proportional to the form of an inverse Gamma distribution with parameters $\alpha = N/2 = (n-1)/2$ and $\beta = Ns^2/2 = (N/2)(1/N)s^2 = 1/2s^2 = 1/2\sum_{i=1}^{n}(y_i - \overline{\mu}^2)$. The posterior distribution of the variance is now given by

$$\sigma^2|Y \sim Inv-Gamma\left(\frac{N}{2}, \frac{1}{2}\sum_{i=1}^{N+1}(y_i - \overline{\mu})^2\right). \tag{89}$$

Next, given the draw for σ we sample from the posterior of the mean

$$p(\mu|\sigma, Y) \sim \mathcal{N}\big(\overline{\mu}, \sigma^2/N\big). \tag{90}$$

We observe that Equation (90) captures the dependence of the posterior mean on the size of the available dataset. As N becomes larger, the variance of μ becomes lower as a result of the smaller uncertainty around its true value which in turn stems from the more available data. To obtain an accurate approximation of the posterior distribution, we sample one million draws from Equations (89) and (90) to generate every time one value for μ and one value for σ. Now, for an investor who considers parameter uncertainty to sample from the predictive posterior we create a return value for each pair (μ, σ^2) (if we create $1,000,000$ pairs of μ and σ from Equations (89) and (90), we will generate $1,000,000$ return values, one for each pair). These returns $(R_1, \ldots, R_N)$ are the inputs to the Monte Carlo simulations we run to obtain the optimal weights.

The difference between an agent who considers parameter uncertainty and one who ignores it, lies in the way the returns are modelled; the latter creates new samples by drawing from a distribution with fixed parameter values while the former uses each time one of the pairs (μ, σ^2) generated by the sampling procedure.

D.2 Proof of Lemma 2 (Parameter Uncertainty with Predictable Returns)

We present the Bayesian framework for the case of returns predictable through the dividend yield. Under the assumption of normality and working with the compact form of the VAR as in Equation (21), the likelihood of B, Σ given X, Z, where Σ is the residual positive-definite covariance matrix, takes the form of

$$\begin{aligned}\mathcal{L}(B,\Sigma|X,Z) &= \frac{1}{\sqrt{((2\pi)^k|\Sigma|)^n}}\exp\Big\{-\frac{1}{2}\sum_{i=1}^{n}(X_i-B'Z_i)'\Sigma^{-1}(X_i-B'Z_i)\Big\}\\ &= \frac{1}{\sqrt{((2\pi)^k|\Sigma|)^n}}\exp\Big\{-\frac{1}{2}tr(X-BZ)(X-BZ)'\Sigma^{-1}\Big\}\\ &\propto |\Sigma|^{(-n/2)}\exp\Big\{-\frac{1}{2}tr(X-BZ)(X-BZ)'\Sigma^{-1}\Big\}\\ &= |\Sigma|^{(-n/2)}\exp\Big\{-\frac{1}{2}tr[S+(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1}\Big\},\end{aligned} \tag{91}$$

where tr is the the trace function. The last equality follows from $(X-BZ)'(X-BZ) = (X-\hat{B}Z)'(X-\hat{B}Z)+(B-\hat{B})'ZZ'((B-\hat{B})) = S+(B-\hat{B})'ZZ'((B-\hat{B}))$. A suitable uninformative prior given independence between B and Σ is the Jeffreys prior given by

$$\begin{aligned}p(B,\Sigma) &= p(B)p(\Sigma)\\ &\propto |\Sigma|^{-(m+1)/2},\end{aligned} \tag{92}$$

with $p(B)$ a constant. Now, combining the prior in Equation (92) with the likelihood in Equation (91) we derive the joint posterior for B and Σ

$$\begin{aligned}p(B,\Sigma|Z,X) &\propto |\Sigma|^{-(m+1)/2}|\Sigma|^{(-n/2)}\exp\Big\{-\frac{1}{2}tr[S+(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1}\Big\}\\ &= |\Sigma|^{-(n+m+1)/2}\exp\Big\{-\frac{1}{2}tr[S+(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1}\Big\}.\end{aligned} \tag{93}$$

We notice that Equation (93) can be written in a similar form to the expression for the i.i.d. case as

$$p(B,\Sigma|Z,X) = p(B|\Sigma,Z,X)\times p(\Sigma|Z,X), \tag{94}$$

which is equal to

$$p(B,\Sigma|Z,X) \propto |\Sigma|^{-(n+m+1)/2} \exp\left\{ -\frac{1}{2} tr[(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1} \right\} \times \exp\left\{ \frac{1}{2} tr\ S\Sigma^{-1} \right\}. \tag{95}$$

Chatfied and Collins (2013) show that we can split Equation (95) as

$$p(B|\Sigma,Z,X) \propto |\Sigma|^{-k/2} \exp\left\{ -\frac{1}{2} [(\beta-\hat{\beta})'\Sigma^{-1} \otimes Z'Z(\beta-\hat{\beta})] \right\}, \tag{96}$$

where $\beta = (\beta_1,\ldots,\beta_m)$ is the matrix of regression coefficients ($\hat{\beta}$ are their estimates), $\otimes$ is the Kronecker product operator and $\nu = n-k+m+1$ and

$$p(\Sigma|Z,X) \propto |\Sigma|^{-\nu/2} \exp\{tr\ S\Sigma^{-1}\}, \tag{97}$$

where $S = (X-\hat{B}Z)'(X-\hat{B}Z)$. It can be proved (Tiao and Zellner, 1964; Zellner, 1996) that the conditional posterior for B is in the form of a multivariate normal density function with mean $\hat{\beta}$ and covariance $\Sigma^{-1} \otimes Z'Z$ while the posterior predictive for Σ in Equation (97) is distributed as

$$\Sigma \sim \mathcal{W}^{-1}((X-Z\hat{B})'((X-Z\hat{B}), T-n-1). \tag{98}$$

In Equation (98), $\mathcal{W}^{-1}$ represents the inverse Wishart distribution with parameters the variance-covariance matrix scaled estimator and $T-n-1$ degrees of freedom. In order to facilitate and speed up the sampling procedure we inverse this relationship to obtain the distribution of the inverse variance covariance matrix. This follows now the distribution

$$\Sigma^{-1} \sim \mathcal{W}([(X-Z\hat{B})'((X-Z\hat{B})]^{-1}, T-n-1). \tag{99}$$

with the same degrees of freedom and the sampling parameter equal to the inverse of the variance covariance matrix estimator.

To sample from the posterior distribution we use a standardized procedure. We first sample from $p(\Sigma^{-1}|X)$ (the variance-covariance matrix) and then given this draw we from the $p(vec(B)|\Sigma^{-1},X) = \mathcal{N}(vec(\hat{B}), \Sigma^{-1} \otimes Z'Z)$ (the AR matrix and the constant coefficients). Given these draws we simulate forward the VAR to obtain future stock return paths. This specification captures the uncertainty in stock returns' forecasts since the VAR parameters are not taken as the true ones as in the case where parameter uncertainty is ignored. An investor who uses the latter approach simulates forward the VAR based on fixed parameters obtained by the calibration using observed data.

In the next step of the sampling procedure we calculate the mean and variance of the first two moments of the return paths generated in step one. Based on these statistics we generate draws from the return posterior distribution for the stock returns and their variance which are now normally distributed. Each draw represents a quarterly return and variance. In our case, given the 40-year horizon, we sample $40 \times 4 = 160$ points, which are the inputs to the dynamic programming algorithm to solve the DA portfolio problem.

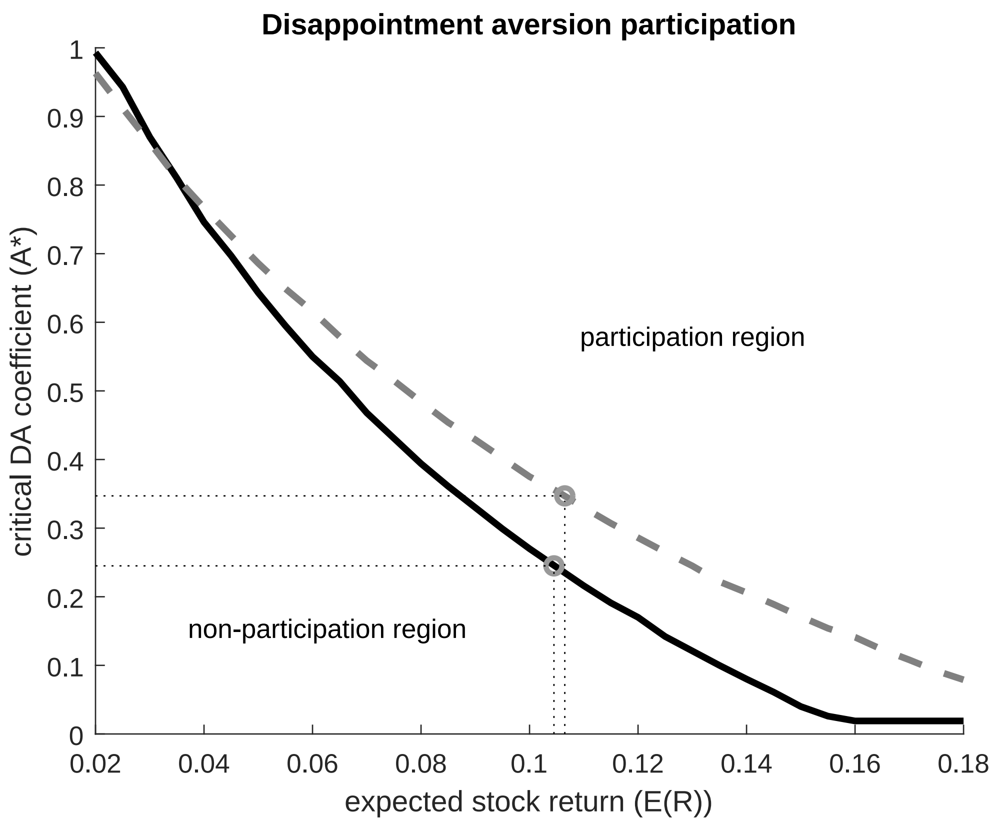

Figure 1: Stock market participation/nonparticipation regions with DA preferences. The graph shows how the expected level of stock returns (stated annually) affects the critical level of the DA coefficient (A^*). Two lines are presented: the solid one corresponds to the critical DA coefficients for the data set used in our study (1934–2016), and the dashed line plots the critical DA values for the data set used in Ang et al. (2005). The gray squares represent the critical DA level (A^*, which induces nonparticipation), which corresponds to the historical mean of the equity return for the two data samples.

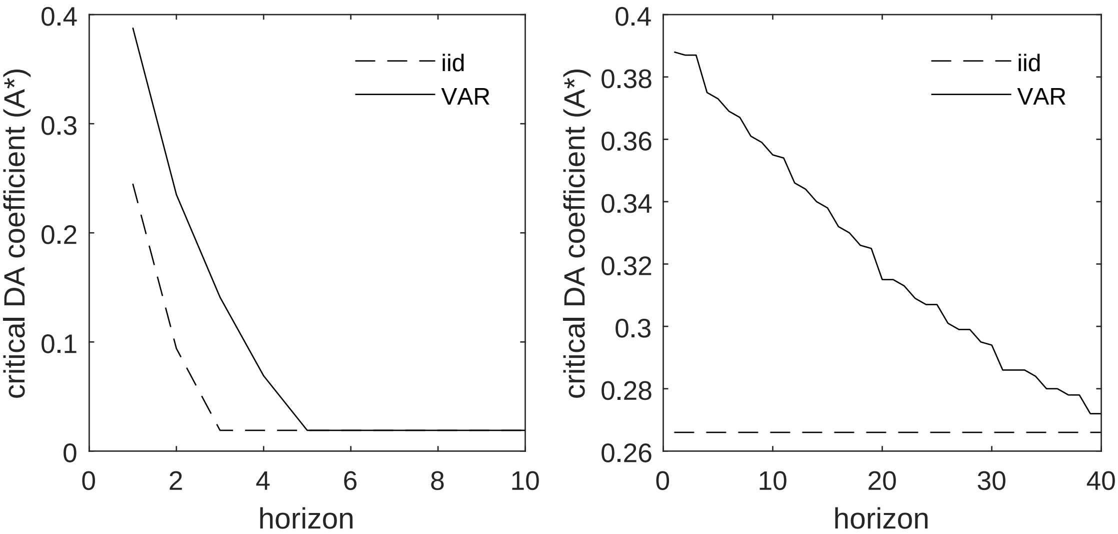

Figure 2: Critical DA level (A^*) induces nonparticipation in the stock market for a buy-and-hold investor (left graph) and a dynamic investor (right graph). The dashed line corresponds to the case of i.i.d. returns (normality and nonpredictability), and the solid line corresponds to the case of predictable returns. Investors invest in the stock market when their DA coefficient lies in the area above the lines. To display the graphs more clearly, the one on the left (buy-and-hold) plots the A^* for a period up to 10 years, as beyond that point A^* remains constant and very close to zero.

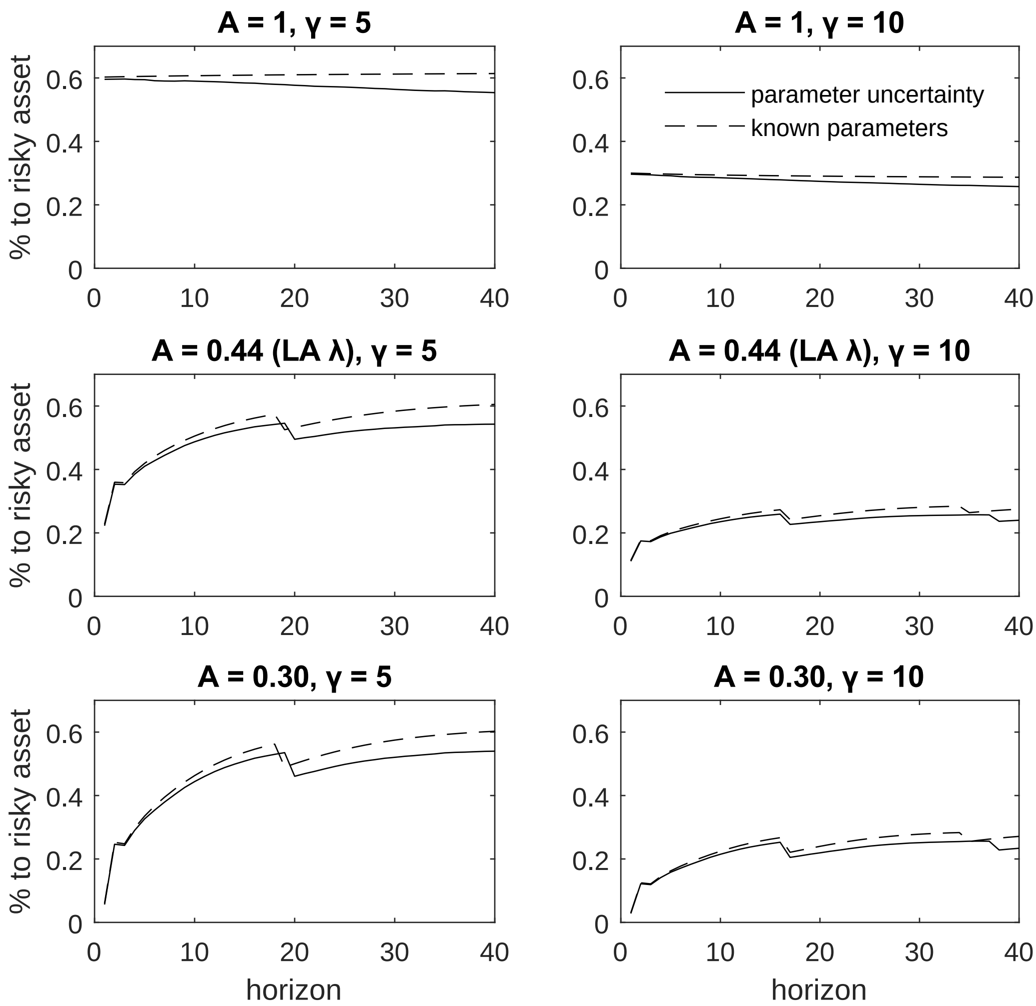

Figure 3: Optimal portfolio allocation to the risky asset for an investor who follows a buy-and-hold investment strategy, uses the i.i.d. return generator, and either incorporates (solid line) or ignores (dashed line) uncertainty in model parameters. The investor in the top row uses a CRRA (i.e., power) utility function with two levels of risk aversion, whereas the other two cases (middle and bottom rows) make use of the DA utility function with two different values for the DA coefficient. $A = 0.44$ is equivalent to the value of the loss aversion (LA) parameter calculated in Tversky and Kahneman (1992), that is, $DA = 1/\lambda = 0.44$. We observe that a DA investor holds a significantly different portfolio from one who uses a power utility function.

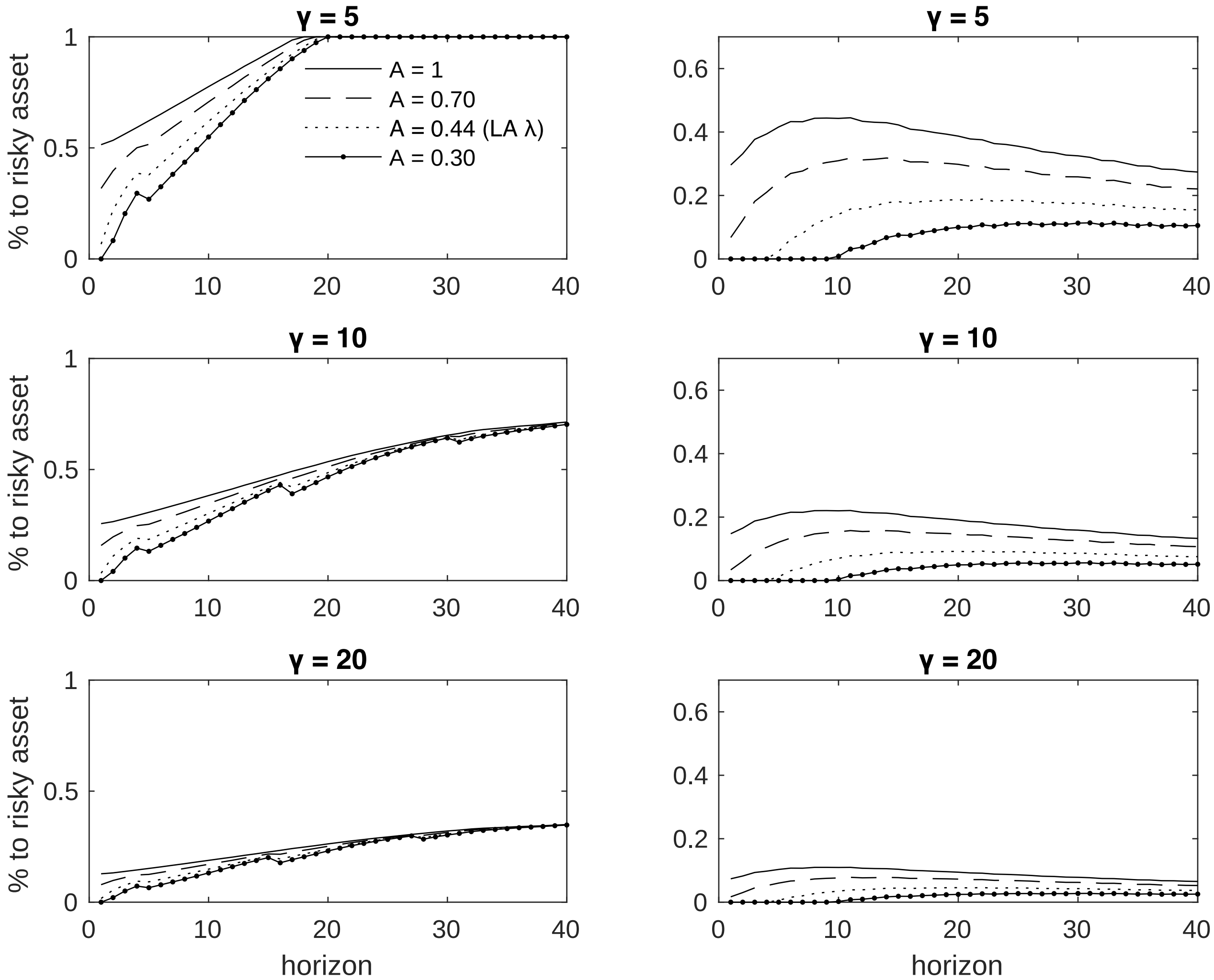

Figure 4: Optimal portfolio allocation to equities for different horizons when the VAR is used to forecast equity returns. The investor follows a buy-and-hold strategy by choosing the portfolio allocation to the risky asset in the beginning of the investment period. $A = 0.44$ is equivalent to the value of the loss aversion (LA) parameter calculated in Tversky and Kahneman (1992), that is, $DA = 1/\lambda = 0.44$. The graphs on the left column ignore parameter uncertainty, whereas those on the right account for this. Three levels of risk aversion and four levels of disappointment aversion are represented.

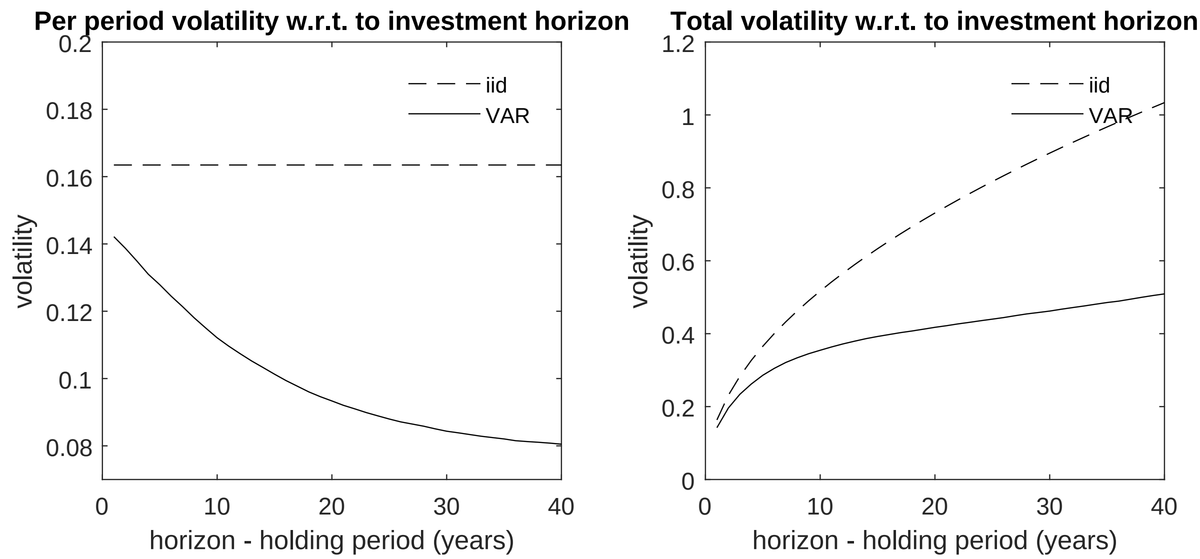

Figure 5: Evolution of per-period and long-term volatility for the risky asset. The dotted line corresponds to the case of an investor who models returns as i.i.d., whereas the solid line shows the volatility for an investor who uses the VAR to forecast equity returns.

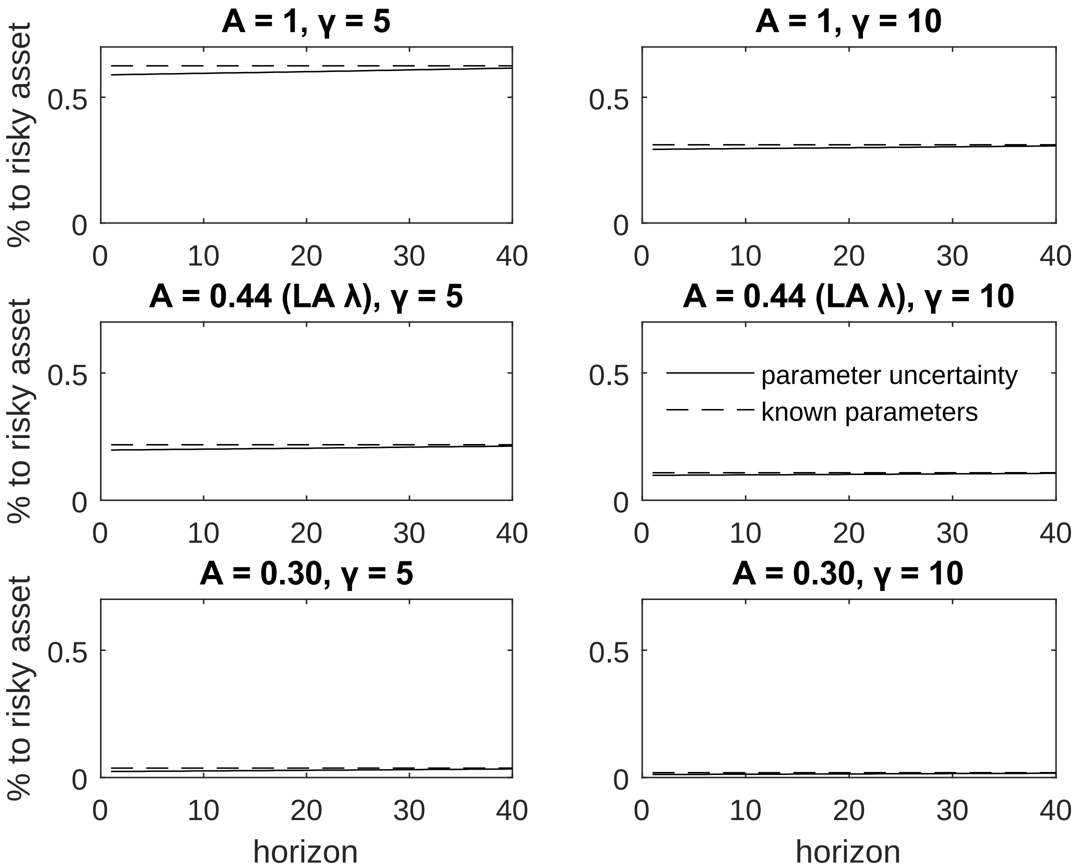

Figure 6: Dynamic portfolio allocation between the risky and the riskless asset for an investor who uses the i.i.d. return generator for the risky asset. The objective of this exercise is to show how the portfolio allocation to the risky asset changes for an investor who acknowledges parameter uncertainty (solid line) compared with one who ignores it (dashed line) and holds the same portfolio throughout the investment horizon. $A = 0.44$ is equivalent to the value of the loss aversion (LA) parameter calculated in Tversky and Kahneman (1992), that is, $DA = 1/\lambda = 0.44$.

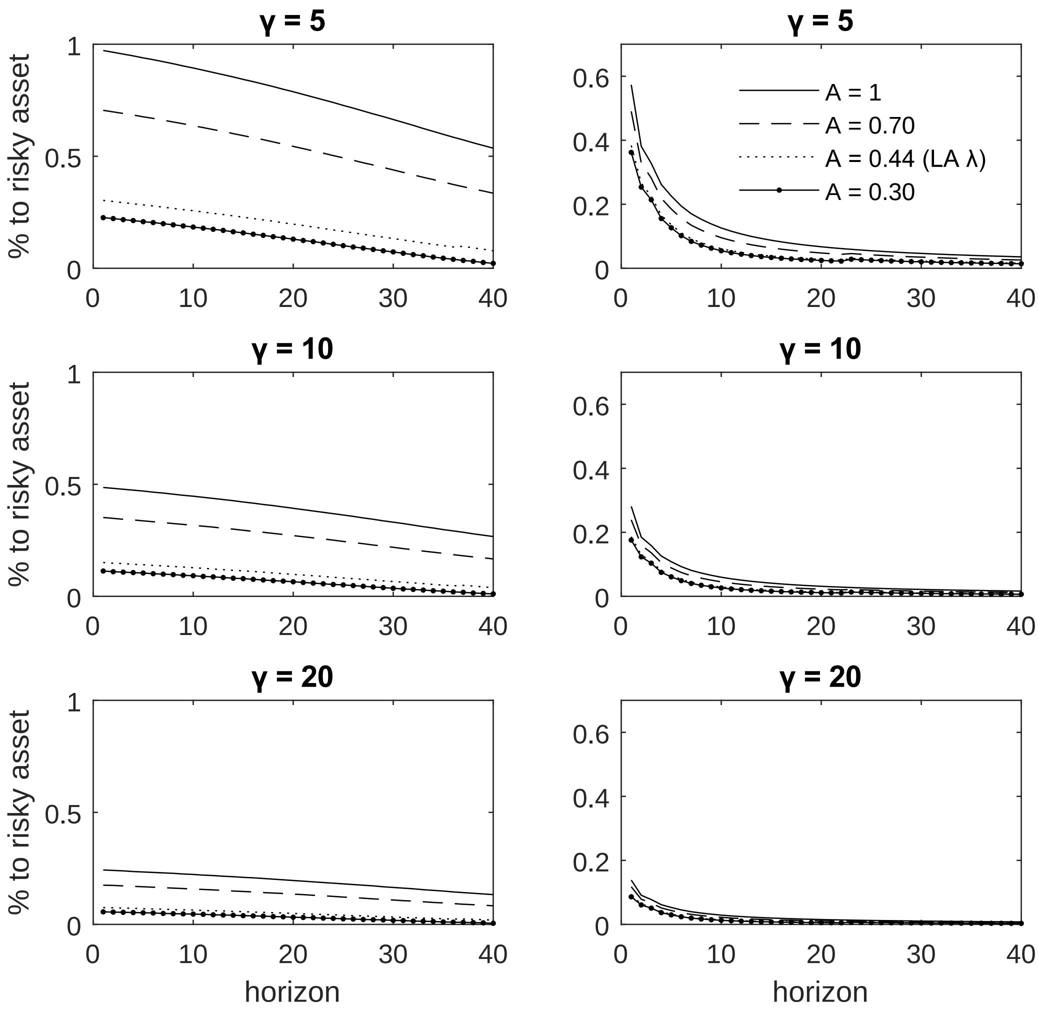

Figure 7: Optimal portfolio allocation at different time horizons for an investor who follows a dynamic reallocation using the VAR to forecast returns. The left columns report results when parameter uncertainty is ignored, whereas the one on the right accounts for parameter uncertainty. Each line corresponds to a different level of the DA coefficient (A) as follows: solid line, A = 1; dashed line, A = 0.70; dotted line, A = 0.44; solid/dotted line, A = 0.30. $A = 0.44$ is equivalent to the value of the loss aversion (LA) parameter calculated in Tversky and Kahneman (1992), that is, $DA = 1/\lambda = 0.44$.

Table 1
Summary statistics

	S&P 500	3-month T-bill	Excess Return
		Annualized	
mean	0.1045	0.0344	0.0695
stdev	0.1625	0.0088	0.1644
		Quarterly	
mean	0.0251	0.0085	0.0166
stdev	0.0817	0.0044	0.0822

S&P 500 and T-bill summary statistics annualized. Excess return is calculated by subtracting the 3-month T-bill rate from the value of the S &P 500 for the same period.

Table 2
Parameter estimates for the Data Generating Process (VAR)

Parameter	With predictability	Without predictability
c_1	0.1222	0.0128
	(0.0173)	(0.0178)
c_2	-0.0004	-0.0317
	(0.0119)	(0.0150)
b_{11}	0.0259	0.0
	0.1176	–
b_{12}	0.0220	0.0
	(0.1354)	–
b_{21}	-0.7068	0.0
	(0.0807)	–
b_{22}	0.9978	0.9932
	(0.0929)	(0.0912)
σ_{11}	0.0850	0.0856
	(0.0037)	(0.0042)
σ_{22}	0.0408	0.0752
	(0.0017)	(0.0029)
ρ	-0.5216	-0.2980
	(0.0021)	(0.0028)

This table shows VAR estimation and the corresponding standard errors of the parameters for the two systems (predictability/no predictability). We utilized maximum likelihood (MLE) to calculate the model in Equation (20). For the nonpredictability system, the autoregressive coefficient matrix is set to zero, whereas, when we account for predictability in returns, all four coefficients are free to vary without restrictions. Parentheses include the standard errors of the estimated coefficients. The S&P 500 index and dividend yield quarterly data for the period January 1934 to September 2016 are used in our calculations.