Asset Liquidity in Monetary Theory and Finance: A Unified Approach*

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Abstract

Economists often say that certain types of assets, e.g., Treasury bonds, are very 'liquid'. Do they mean that these assets are likely to serve as media of exchange or collateral (a definition of liquidity often employed in monetary theory), or that they can be easily sold in a secondary market, if needed (a definition of liquidity closer to the one adopted in finance)? We develop a model where these two notions of asset liquidity coexist, and their relative importance is determined *endogenously* in general equilibrium: how likely agents are to visit a secondary market in order to sell assets for money depends on whether sellers of goods/services accept these assets as means of payment. But, also, the incentive of sellers to invest in a technology that allows them to recognize and accept assets as means of payment depends on the existence (and efficiency) of a secondary market where buyers could liquidate assets for cash. The interaction between these two channels offers new insights regarding the determination of asset prices and the ability of assets to facilitate transactions and improve welfare.

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1 Introduction

Asset liquidity recently has been front and center in the fields of monetary economics and finance. Strikingly, while these two strands of the literature agree that asset liquidity is essential for the study of a number of important topics (such as asset pricing, the implementation of monetary policy, and others), they employ *different* definitions of the term. In monetary theory, liquidity is typically an attribute of the asset itself, and it refers to how easily it can be used to purchase consumption. In finance, liquidity is typically an attribute of the (secondary) market where the asset trades, and it refers to the speed with which an individual can sell the asset, if needed.[1] In reality, both of these approaches are relevant. Sometimes agents use assets directly, either as media of exchange or collateral, to purchase goods and services from sellers, as is typically assumed in monetary theory. Other times, agents with a consumption need sell (or, as we often say, 'liquidate') assets in a secondary market, and then use the cash to purchase goods or services; this notion of liquidity is closer to the one adopted by finance.

This discussion raises a number of questions. What determines whether assets can be used directly as means of payment in transactions between buyers and sellers or the buyer/asset holder must first liquidate them in a secondary asset market, and then use the cash to purchase goods and services? Similarly, when economists say that certain assets, such as Treasury bonds, are "very liquid" do they mean that it is easy to use them as means of payment to purchase commodities or that it is easy to sell them in a secondary market? (An analogous question arises for assets that are considered illiquid, e.g., municipal bonds.) Last, but not least, are these details regarding the different liquidity aspects of assets important for the determination of asset prices and for their ability to facilitate transactions and improve welfare?

To answer these questions, one must employ a model that encompasses both of these notions of liquidity. Developing such model is the first main contribution of this paper. We build on the Lagos and Wright (2005) framework, where certain frictions, such as anonymity and imperfect commitment, impede trade in commodity markets and make a medium of exchange or collateral necessary. Fiat money helps bypass these frictions by serving as means of payment. Alongside money, a real asset can also potentially serve as a facilitator of trade. However, due to asymmetric information regarding the quality of the asset, only a fraction of sellers recognize and accept it in transactions (but all sellers accept money). We determine this fraction endogenously by allowing sellers to invest in information about the asset. As long as some sellers do not accept assets as payment, the buyers who are "matched" with them cannot use assets directly to buy goods. But even these buyers can benefit from the asset's 'liquidity', as they can visit a secondary market and sell their assets for money. Following Duffie et al. (2005), we assume that

[1] This argument is also highlighted by Lagos (2008). For examples of papers in the first strand of the literature, see Lagos, Rocheteau, and Wright (2017) and the references therein; for examples of papers in the second strand of the literature, see Duffie, Gârleanu, and Pedersen (2005) and the references therein.

this market is an over-the-counter (OTC) market, characterized by search and bargaining.[2]

Therefore, in our model, like in reality, sometimes assets compete *with money* as direct media of exchange, and some other times they must be liquidated *for money* in a secondary asset market, upon the arrival of a consumption opportunity. To fix ideas, we will refer to the former type of liquidity as *direct* asset liquidity and to the latter as *indirect*. Our paper not only provides a theory where both of these notions of liquidity coexist, but one where their relative importance is determined *endogenously* as a function of two fundamental parameters: i) The information cost that sellers must incur in order to recognize and accept assets in transactions; and ii) The efficiency of matching in the secondary OTC asset market.

Our model extends that of Lester, Postlewaite, and Wright (2012) (henceforth, LPW). Following that paper, we adopt an environment where sellers of goods who are not informed about the asset simply refuse to accept it in trade.[3] An important contribution of LPW is to endogenize the decision of sellers to invest in information/technology to distinguish high and low quality versions of certain assets. Hence, LPW is a model of *direct* liquidity: assets are liquid only to the extent that sellers invest in the information that allows them to (recognize and) accept them as media of exchange. Consequently, and in the authors' own words, "in any situation where buyers and sellers are asymmetrically informed about the values of assets, exchange is hindered". But this statement seems incomplete. If a seller turns down a buyer's, say, T-Bills because she does not recognize them, that does not mean that no trade can take place: the buyer could still go to the secondary market for Treasuries—where, importantly, recognizability is not an issue—sell some bonds and return to the seller with her preferred method of payment, i.e., cash.

The present paper adds precisely this channel, i.e., it adds *indirect* asset liquidity. However, it is imperative to highlight that our model of direct *and* indirect liquidity is greater than the sum of its parts because, in general equilibrium, the degree of direct asset liquidity affects and is affected by the degree of indirect liquidity: how likely an agent is to "visit" the secondary market to liquidate assets, crucially depends on whether the seller of the goods/services she wishes to purchase will accept these assets as payment. Vice versa, the incentive of a seller to invest in the technology that allows her to recognize assets is affected by the existence (and efficiency) of a secondary market where the buyer can liquidate assets for cash. Our model studies the interaction between these two channels, and delivers a number of new insights.

We start with a model where the fraction of sellers who accept assets is exogenous. Agents make their portfolio choice between money and the real asset without knowing whether the seller they will meet accepts assets or not. Ex post, some agents match with sellers who accept

[2] This is arguably an empirically relevant choice. In the the United States, Neklyudov and Sambalaibat (2015) report that the fraction of aggregate asset trade volume that took place in OTC markets was around 87% in 2010.

[3] As LPW explain, this is technically convenient because it helps avoid bargaining under asymmetric information, which is significantly more complicated. Rocheteau (2011) and Li, Rocheteau, and Weill (2012) study liquidity-related questions in models where assets that are not recognized may be partially accepted in trade.

the real asset, and some with sellers who do not. Once this idiosyncratic uncertainty has been resolved, an obvious motive for trade arises. The latter agents need more money because that is all they can use as means of payment. The former agents are happy to exchange some money for real assets because, in their hands, these two objects are equally effective media of exchange. These are precisely the types of trades that the secondary OTC market allows to materialize.

In this environment agents are willing to pay a liquidity premium for the asset if its marginal unit: i) helps them acquire more goods by serving as payment (direct liquidity), and/or ii) helps them acquire additional money in the OTC, thus, relaxing a binding cash constraint (indirect liquidity). Therefore, the asset price will include a liquidity premium as long as its supply is not too plentiful, and we provide a detailed characterization of the parameter space for which each type of liquidity is relevant. This has important consequences for the effect of monetary policy on asset prices. For instance, if asset supply is low and inflation is intermediate, the asset can be valued both for its direct and indirect liquidity. Within this region, a rise in inflation not only increases the asset price (an established result in the literature), but it does so with a high elasticity. If inflation increases further, real money balances are depressed. Eventually, we enter a region where even though the agent's asset holdings are low, they are high enough to purchase all the real money balances available in the OTC market. That is to say, the indirect liquidity motive vanishes, and this lowers the elasticity of the asset price with respect to inflation.

We also study how some key equilibrium variables depend on the fraction of sellers who accept assets, call it λ. First, the asset price is increasing in λ because agents are willing to pay a higher liquidity premium for an asset more likely to serve as a medium of exchange. Also, the volume of trade in the OTC market is hump-shaped in λ. To see why, consider the extremes, $\lambda = 1$ or 0. When all sellers accept assets in transactions, the role of the OTC market is depleted, and the volume of trade is zero. On the other extreme, when no sellers accept assets, every agent would like to sell assets for money in the OTC, but no one is willing to supply it, because money is the sole medium of exchange in this economy. Again, the end result is no trade. Thus, the OTC trade volume is zero at the extremes, and positive for intermediate values of λ.

The next step is to endogenously determine the fraction of sellers who accept assets in trade. To do so, we analyze the best response of a typical seller who believes that a fraction λ of (other) sellers accept assets. If λ is large, that seller has a lot to lose by not acquiring information: a high λ implies that agents can use their assets as means of payment often, which depresses money balances and hurts sellers who chose to not acquire information and only accept money. On the other hand, when λ is high, the agents who need to sell assets for money in the OTC market are few, and those who are willing to provide money (because they can use assets for exchange) are many: with market tightness in their favor, agents who seek to boost their money holdings are likely to succeed. In sum, a higher λ induces agents to carry less money *ex ante*, but it implies that agents who turn out to need money for trade are more likely to acquire it *ex post* in the OTC market. The first force encourages sellers to acquire information, thus, promoting coordination and corner equilibria.[4] The second force discourages sellers from acquiring information and tends to generate *stable* interior equilibria, where some agents use assets directly as payment, and some others visit the OTC to liquidate assets for money. We find interior equilibria especially interesting, because they are arguably more empirically relevant.

With these opposing forces at work, multiplicity of equilibrium can easily arise. We provide a detailed discussion of all possible equilibria, and show that a stable interior equilibrium will exist under fairly general conditions. Around this equilibrium, an increase in the information cost that sellers must incur to recognize assets leads to a lower equilibrium λ. Moreover, an increase in the efficiency of matching in the OTC market reduces the measure of sellers who acquire information: a more efficient OTC market allows buyers to take advantage of the *indirect* liquidity properties of the asset, thus reducing sellers' incentives to invest in the information that makes assets *directly* liquid. In a sense, the the asset's indirect liquidity through the OTC market serves as a substitute to its direct liquidity.

Our model also has some policy relevant implications. Oftentimes, central banks and financial regulators (are concerned about and) wish to "improve asset liquidity". But which liquidity? Should authorities boost direct liquidity, say, by subsidizing the information cost that sellers incur to recognize assets? Or indirect liquidity, say, by improving efficiency in the OTC market (e.g., by promoting a more efficient interdealer market)? A surface-level examination suggests that these two alternatives are equivalent, and one should not worry about the details.[5] However, we show that a decrease in the information cost and an increase in the OTC matching efficiency generate multiple, opposing channels and have very different effects on welfare.

A decrease in the information cost increases the number of meetings where assets serve as means of payment and, typically, enhances welfare (although possible exceptions are discussed). In contrast, an increase in OTC market efficiency is likely to hurt welfare. The intuition behind this result lies on the interaction of two effects. First, as discussed earlier, a more efficient OTC market crowds out the asset's direct liquidity, i.e., it reduces the fraction of sellers who accept assets. Second, a more efficient OTC market depresses money balances, since agents expect it will be easier to acquire money in the OTC market if they need it. Taken together, a more efficient OTC market generates a larger number of meetings where only money can serve as means of payment and, at the same time, reduces the amount of money that agents carry in equilibrium. We highlight cases where this effect is so powerful that the economy would be better off if the secondary market did not exist altogether. Thus, our model suggests that the authorities should promote the direct liquidity of assets. This may include improving financial literacy among economic agents and reducing the information asymmetry about asset returns.

[4] When this force prevails, a seller who believes that all other sellers accept assets, i.e., $\lambda = 1$, finds it optimal to also accept assets, thus reinforcing $\lambda = 1$ as an equilibrium. A similar argument applies to the case with $\lambda = 0$.

[5] After all, our results so far indicate that direct and indirect liquidity are substitutable.

1.1 Literature Review

The present paper is related to a large and growing literature that has pointed out the importance of asset liquidity for the determination of asset prices. Examples of such papers include Geromichalos, Licari, and Suarez-Lledo (2007), Lagos and Rocheteau (2008), Lagos (2010), Nosal and Rocheteau (2013), Andolfatto and Martin (2013), Andolfatto, Berentsen, and Waller (2013), Venkateswaran and Wright (2014), Rocheteau and Wright (2013), Hu and Rocheteau (2015), Jung and Lee (2015), Geromichalos, Lee, Lee, and Oikawa (2016), and Jung and Pyun (2016), among many others. In these papers, assets are 'liquid' because they can facilitate transactions in frictional decentralized markets, by serving directly as means of payment or collateral.

The closest paper to ours is Lester et al. (2012) (LPW), who extend the aforementioned literature by endogenizing the measure of sellers who accept assets. We add to this work by explicitly modeling a secondary market, where agents who cannot use assets as means of payment can boost their money holdings by selling these assets. Incorporating this 'indirect liquidity' channel amounts to much more than just adding an empirically relevant ingredient to the LPW framework, as the interaction between direct and indirect liquidity offers a number of new insights.[6] Except from the differences in asset prices (which are now affected by the structure of the OTC market), our novel indirect liquidity channel dramatically changes the properties of equilibrium. In LPW, a seller's profit is always increasing in the fraction of other sellers who accept assets, making corners the only stable equilibria. Here, that channel is also present, but now a seller who chooses to stay uninformed may be better off when more sellers acquire information, because she will meet an agent who is more likely to have boosted her money holdings in the OTC market. This new force tends to generate *stable interior* equilibria, where only a fraction of sellers choose to accept assets in trade.[7] Finally, our model delivers a number of surprising welfare-related results discussed earlier. For instance, the introduction of indirect liquidity will crowd out direct liquidity and could ultimately hurt welfare.

Our model is related to a number of recent papers that exploit the idea of indirect liquidity, i.e., the fact that assets can be sold in a secondary market upon the arrival of a liquidity need. Examples include Geromichalos and Herrenbrueck (2016), Mattesini and Nosal (2016), Berentsen,

[6] The empirical relevance of the indirect liquidity notion is best reflected in the words of Brian Roseboro, the Assistant Secretary of the US Treasury for the period 2001-2004, who states that "A deep, liquid [...] secondary market serves our goal of lowest-cost financing for the taxpayer by encouraging more aggressive bidding in the primary market." ("A Review of Treasury's Debt Management Policy", June 3, 2002, available at https://www.treasury.gov/press-center/press-releases/Pages/po3149.aspx). The official indicates that investors are willing to pay a higher liquidity premium for Treasury bonds (in the primary market) if they expect to be able to sell them easily in the secondary market, a narrative that matches perfectly with our notion of indirect liquidity.

[7] We find interior equilibria interesting both from a theoretical standpoint and because they are arguably more empirically relevant. Our reading of LPW is that the authors also agree with this assessment. But since in that paper the interior equilibrium is unstable, the authors explore an extension of the model, where the information cost is different for each seller. For certain parametric specifications of the distribution of costs among sellers, LPW can generate a stable interior equilibrium. For more details, see Section 4.2.

Huber, and Marchesiani (2014, 2016), Han (2015), Herrenbrueck and Geromichalos (2017), Herrenbrueck (2019), and Madison (2019). In these papers, agents visit the secondary market to sell assets because sellers never accept them as means of payment. Hence, this literature imposes a *cash-in-advance constraint*, dictating that only money can be used as means of payment. In our model, agents visit a secondary market to liquidate assets only if the seller they matched with refuses to accept assets, and whether sellers accept assets or not is determined *endogenously*. Hence, one can see the present paper as one that provides micro-foundations for indirect asset liquidity, and we view this as a significant contribution in itself.

Finally, our work is related to the literature initiated by the pioneering work of Duffie et al. (2005), which studies how bargaining and search frictions in OTC financial markets affect asset prices and trade. Examples of such papers include Vayanos and Weill (2006), Weill (2007), Lagos and Rocheteau (2009), Uslu (2016), Bethune, Sultanum, and Trachter (2016) and Chang and Zhang (2018). Our paper differs from these papers because it introduces an OTC market into a monetary model (where assets also have direct liquidity properties); hence, in our model agents visit the OTC to sell assets because they need money, while in these papers agents trade because they differ in their valuation for the asset. Lagos and Zhang (2015) also consider an environment where gains from trade arise due to differences in asset valuation, but that model is a monetary one (hence, closer to ours) since agents who wish to buy assets must pay with money.

2 The Model

Time is discrete and continues forever. Each period consists of three sub-periods, where different economic activities take place. In the first sub-period agents trade in an Over-The-Counter asset market, characterized by search and bargaining, as in Duffie et al. (2005). We dub this market the OTC. In the second sub-period agents trade in a decentralized commodity market, which we dub the DM. In the third sub-period agents trade in a centralized, competitive market, henceforth referred to as the CM. Before going to the details, we offer an intuitive description of the role played by each market. The CM is the typical settlement market of Lagos and Wright (2005), where agents settle their old portfolios and choose new ones. The DM is a decentralized market characterized by anonymity and imperfect commitment, where agents meet bilaterally and trade goods and services; this can include the retail market, the market for investment goods, etc. Crucially, the frictions in the DM make a medium of exchange (henceforth, MOE) or collateral necessary in transactions, and which assets can serve this role will be determined endogenously. Since some agents may only be able to use money as means of payment, the OTC is placed before the DM so that these agents can visit it and rebalance their portfolios, i.e., sell assets for money. One can think of this market as the secondary market for Treasuries, corporate bonds, municipal bonds, etc.

Agents live forever and discount future between periods at rate $\beta \in (0,1)$. There are two types of agents, buyers and sellers, distinguished by their roles in the DM. Each type's measure is normalized to 1. Buyers consume in the DM and CM and supply labor in the CM; sellers produce in the DM and consume and supply labor in the CM. All agents can transform one unit of labor in the CM into one unit of the CM good, which is the *numeraire*. The preferences of buyers and sellers within a period are given by $\mathcal{U}(X,L,q) = X - L + u(q)$ and $\mathcal{V}(X,L,q) = X - L - q$, respectively, where X denotes consumption of CM goods, L is labor supply in the CM, and q stands for the amount of DM good traded. We assume that u is twice continuously differentiable, with $u' > 0$, $u'(0) = \infty$, $u'(\infty) = 0$, and $u'' < 0$. Let q^* denote the first-best level of trade in the DM, i.e., $\{q^* \equiv q : u'(q^*) = 1\}$. All goods are perishable between periods.

There are two types of assets: fiat money and a one-period real asset. Buyers can purchase any amount of money and the asset at (real) prices φ and ψ in the CM, respectively.[8] The supply of money is controlled by a monetary authority, and follows the rule $M_{t+1} = (1+\mu)M_t$, with $\mu > \beta - 1$. New money is introduced if $\mu > 0$, or withdrawn if $\mu < 0$, via lump-sum transfers to buyers in the CM. Money has no intrinsic value, but it possesses the standard properties that make it an acceptable MOE in the DM, most notably it is *recognizable* by everyone in the economy. The supply of the real asset is fixed and denoted by $A > 0$. Each unit of the asset purchased today delivers one unit of numeraire in next period's CM.

Moving on to the DM, the important question is which assets can serve as means of payment in that market. Following LPW, we assume that all sellers (recognize and) accept money but, due to asymmetric information about the quality of real assets, only a fraction of sellers accept them in trade. More precisely, an asset can obtain a high or a low value and, for simplicity, it is assumed that the low-value asset is completely worthless. As LPW point out, one can think of the low-quality asset as "a bad claim to a good tree" (i.e., a counterfeit) or "a good claim to a bad tree" (i.e., a lemon); for our analysis this distinction does not matter. Buyers can produce the worthless asset at zero cost, and, as a result, sellers optimally choose to never accept an asset that they do not recognize.[9] We refer to sellers who accept only money as type 1 sellers and to those who accept both money and assets as type 2 sellers, and we let $\lambda \in [0,1]$ denote the measure of type 2 sellers. As a starting point, we treat λ as an exogenous parameter, but eventually we endogenize this term (in Section 4), by assuming that sellers can acquire information that allows them to recognize the real asset. To keep things simple, we assume without loss of generality that all buyers meet a seller in the DM (and vice versa). Within each match the terms of trade are determined by Kalai bargaining, with $\eta \in (0,1)$ denoting the buyer's bargaining power.

[8] In this model, sellers will never choose to hold assets, as long as they are priced at a liquidity premium: a seller's identity is permanent, so why would she pay a liquidity premium if she knows that she will never have a liquidity need (in the DM)? As a result, the interesting portfolio choices are made by the buyers.

[9] This setup allows us to capture the idea that an asset's recognizability affects its ability to serve as a MOE, without making the analysis cumbersome. Basically, as explained in footnote 3, these assumptions help us avoid dealing with bargaining under asymmetric information, which is significantly more complicated.

After buyers have made their portfolio decision in the CM, but before they visit the DM, they discover which type of seller they will match with in the DM. A fraction λ of buyers will meet with type 2 sellers, and the rest will meet with type 1 sellers. For convenience, we will call the former type 2 buyers and the latter type 1 buyers. Clearly, the type of seller with whom a buyer matches in the DM determines which assets she can use as MOE. Buyers who turn out to be of type 1 will not be able to use their asset to buy the DM good, but they can visit the OTC market to sell some assets for cash, and type 2 buyers will be happy to (buy assets and) provide that cash, because for them real assets, as well as money, are acceptable MOE.

It should now be clear that the OTC market allows a more efficient reallocation of liquidity or, alternatively, it allows money to end up in the hands of the agents who value it most (the type 1 buyers). The matching technology in the OTC market is described by the constant returns to scale function $f(x, y) = \alpha \frac{xy}{x+y}$, where x, y are the measures of asset buyers and sellers, respectively, and α is a parameter that measures the matching efficiency. Clearly, here $x = \lambda$ and $y = 1 - \lambda$. Thus, the total number of matches per period is $f(\lambda, 1 - \lambda) = \alpha\lambda(1 - \lambda)$. To keep the analysis as simple as possible, we assume that type 1 buyers make a take-it-or-leave-it (henceforth, TIOLI) offer to type 2 buyers.[10] The timing of events is summarized in Figure 1.

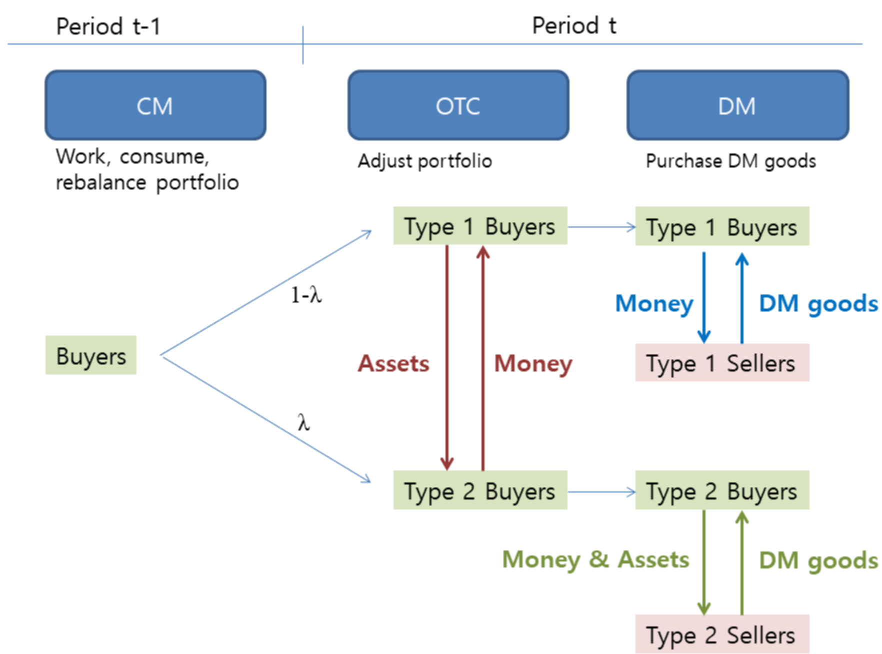


Figure 1: Timing of Events

[10] We choose to give all the bargaining power to type 1 buyers, because this is the interesting case: it is agents who plan to sell the asset 'down the road' (in the secondary market) who are crucial for the determination of the issue price (in the CM). Put differently, if type 2 buyers made the TIOLI offer, the asset would never carry a liquidity premium because of its ability to be sold for cash in the OTC market.

3 Exogenous Asset Acceptance λ

3.1 Value Functions in Subperiods

The value function of a buyer who enters the CM with money and asset holdings (m, a) is given by

$$W(m,a) = \max_{X,L,\hat{m},\hat{a}} \{X - L + \mathbb{E}[\Omega(\hat{m},\hat{a})]\} \tag{1}$$
$$\text{s.t.} \quad X + \varphi\hat{m} + \psi\hat{a} = L + \varphi(m + \mu M) + a,$$

where hats denote next period's choices, and $\mathbb{E}[\Omega(\hat{m},\hat{a})]$ denotes the expected continuation value of a buyer who enters the OTC market with the portfolio $(\hat{m},\hat{a})$. Substituting $X - L$ from the budget constraint allows us to rewrite this value function as

$$W(m,a) = \varphi m + a + \Lambda. \tag{2}$$

As is standard in models that build on Lagos and Wright (2005), W is linear.[11]

Next, the expected value function of a buyer who enters the OTC market with the portfolio (m, a) is given by

$$\mathbb{E}[\Omega(m,a)] = (1-\lambda)\Omega_1(m,a) + \lambda\Omega_2(m,a), \tag{3}$$

where Ω_i represents the value function in the OTC market for a buyer of type $i = \{1, 2\}$. Letting χ denote the units of asset that the type 1 buyer transfers to the type 2 buyer in the OTC, and p the (dollar) price per asset, we can write

$$\Omega_1(m,a) = \alpha\lambda V_1(m + p\chi, a - \chi) + (1-\alpha\lambda)V_1(m,a), \tag{4}$$
$$\Omega_2(m,a) = \alpha(1-\lambda)V_2(m - p\chi, a + \chi) + [1 - \alpha(1-\lambda)]V_2(m,a), \tag{5}$$

where $V_i(m,a)$ denotes the value function of a type i buyer who enters the DM with portfolio (m, a). The interpretation of the OTC value functions is straightforward. If the buyer turns out to be of type 1 (equation (4)), she will try to sell assets for cash in the OTC market. If she is successful, with probability $\alpha\lambda$, she will sell χ units of the asset and acquire $p\chi$ units of money, where p, χ will be determined through bargaining in the OTC market. If she is unsuccessful, with probability $1-\alpha\lambda$, she will simply continue into the DM with her original money holdings. A similar interpretation applies to equation (5).

[11] It is easy to verify that $\Lambda = \mu M + \max_{\hat{m},\hat{a}} \{-\varphi\hat{m} - \psi\hat{a} + \mathbb{E}[\Omega(\hat{m},\hat{a})]\}$, a term unrelated to the state variables (m, a).

Finally, consider the value function in the DM. We have

$$V_i(m,a) = u(q_i) + \beta W(m - d_i^m, a - d_i^a), \tag{6}$$

where q_i, d_i^m, and d_i^a denote the amount of DM good, money, and real asset, respectively, that change hands in a DM meeting between a seller and a buyer of type i. These terms of trade will be determined in Section 3.2.

3.2 Bargaining in the Decentralized Market

Following Kalai's proportional bargaining solution, we can write the bargaining problem in a type i DM meeting between a seller and a buyer with portfolio (m, a) as

$$\begin{aligned} &\max_{q_i, d_i^m, d_i^a} \{u(q_i) + W(m - d_i^m, a - d_i^a) - W(m,a)\} \\ \text{s.t.} \quad & u(q_i) + W(m - d_i^m, a - d_i^a) - W(m,a) = \tfrac{\eta}{1-\eta}\left[-q_i + W^S(d_i^m, d_i^a) - W^S(0,0)\right], \end{aligned}$$

and the feasibility constraints $d_i^m \leq m$ and $d_i^a \leq a$. Of course, we have $d_1^a = 0$, by assumption. The terms W^S denote the seller's CM value function, which are also linear in both arguments.[12] As is standard, the proportional bargaining solution maximizes the buyer's surplus subject to the constraint that a fixed proportion, $(1-\eta)/\eta$, of this surplus is equal to the surplus of the seller. Exploiting the linearity of W and W^S allows one to further simplify the problem to

$$\begin{aligned} &\max_{q_i, d^m, d^a} \eta \{u(q_i) - q_i\} \\ \text{s.t.} \quad & (1-\eta)u(q_i) + \eta q_i = \varphi d_i^m + d_i^a, \end{aligned} \tag{7}$$

and $d_i^m \leq m$ and $d_i^a \leq a$. The following lemma summarizes the bargaining solution.

Lemma 1. *Define $z(q) \equiv (1-\eta)u(q) + \eta q$ and $m^* \equiv \frac{z(q^*)}{\varphi}$. Then, the solution to the bargaining problem in a type 1 meeting is:*

$$d_1^m(m) = \begin{cases} m^*, & \text{if } m \geq m^* \\ m, & \text{if } m < m^* \end{cases} \tag{8}$$

$$q_1(m) = \begin{cases} q^*, & \text{if } m \geq m^* \\ z^{-1}(\varphi m), & \text{if } m < m^* \end{cases} \tag{9}$$

[12] The proof is similar to the one leading to equation (2), but even simpler, because, as we have already discussed, the seller will never leave the CM with any asset holdings.

and $d_1^a = 0$. The solution to the bargaining problem in a type 2 meeting is:

$$(d_2^m(m,a), d_2^a(m,a)) = \begin{cases} (d_2^{m*}, d_2^{a*}), & \text{if } \varphi m + a \geq z(q^*) \\ (m,a), & \text{if } \varphi m + a < z(q^*) \end{cases} \tag{10}$$

$$q_2(m,a) = \begin{cases} q^*, & \text{if } \varphi m + a \geq z(q^*) \\ z^{-1}(\varphi m + a), & \text{if } \varphi m + a < z(q^*) \end{cases} \tag{11}$$

where (d_2^{m*}, d_2^{a*}) is the set of pairs (d_2^m, d_2^a) that satisfies $\varphi d_2^m + d_2^a = z(q^*)$.

Proof. See the appendix. □

The term $z(q)$ represents the value of real liquid balances that induces the seller to produce the quantity q. The amount of DM good a buyer can afford depends on the amount of liquid assets that she carries, but what asset is liquid depends on the meeting. In type 1 meetings, only money can be used as MOE, while in type 2 meetings both money and the real asset are accepted. The rest is straightforward. If the value of the buyer's liquid assets exceeds $z(q^*)$, she will purchase the first-best quantity q^* and spend an amount of assets equal to $z(q^*)$. On the other hand, if the value of the buyer's liquid assets is below $z(q^*)$, she will hand all of them to the seller, only to obtain an amount of DM good which is lower than q^*.

3.3 Bargaining in the OTC market

Next, consider a meeting between a type 1 buyer with portfolio (m,a) and a type 2 buyer with portfolio $(\tilde{m}, \tilde{a})$ in the OTC market, and recall that the former agent makes a TIOLI offer to the latter. The bargaining problem is given by:

$$\max_{p,\chi} \{V_1(m + p\chi, a - \chi) - V_1(m,a)\}$$
$$\text{s.t.} \quad V_2(\tilde{m} - p\chi, \tilde{a} + \chi) - V_2(\tilde{m}, \tilde{a}) = 0,$$

and the feasibility constraints $\chi \in [-\tilde{a}, a]$ and $p\chi \in [-\tilde{m}, m]$. Since p was defined as the dollar price of one unit of asset in the OTC, $p\chi$ is the total monetary boost that the type 1 buyer can obtain by selling assets. After replacing the V functions from equation (6) and some algebra, we can re-write the OTC bargaining problem as:

$$\max_{p,\chi} \{u(q_1(m + p\chi)) - u(q_1(m)) + \varphi\,[d_1^m(m) - d_1^m(m + p\chi)] + [\varphi p\chi - \chi]\}$$
$$\text{s.t.} \quad [u(q_2(\tilde{m} - p\chi, \tilde{a} + \chi)) - u(q_2(\tilde{m}, \tilde{a}))] + [\rho(\tilde{m}, \tilde{a}) - \rho(\tilde{m} - p\chi, \tilde{a} + \chi)] = \varphi p\chi - \chi, \tag{12}$$

$\chi \in [-\tilde{a}, a]$ and $p\chi \in [-\tilde{m}, m]$, where $\rho(m,a) \equiv \varphi d_2^m(m,a) + d_2^a(m,a)$. These mathematical expressions illustrate economic insights that have been already discussed: surplus in the OTC market is generated as money gets transferred into the hands of the type 1 buyer, who can only use this object as a MOE. In return, the type 1 buyer gives some real assets to the type 2 buyer, which is a great deal since, for the latter agent, the real asset (as well as money) is an acceptable MOE. Of course, under the TIOLI assumption, the net surplus that ends up with the type 2 buyer is zero, as indicated by equation (12). The following lemma summarizes the solution to the bargaining problem in the OTC market.

Lemma 2. *Consider a meeting in the OTC market between a type 1 and a type 2 buyer with portfolios* (m, a) *and* $(\tilde{m}, \tilde{a})$*, respectively. Define the "cutoff" level of asset holdings*

$$\bar{a}(m, \tilde{m}) = \begin{cases} \varphi(m^* - m), & \text{if } m + \tilde{m} \geq m^* \\ \varphi\tilde{m}, & \text{if } m + \tilde{m} < m^* \end{cases}$$

Then, the solution to the bargaining problem in the OTC market is given by

$$\chi(m, \tilde{m}, a) = \begin{cases} \bar{a}(m, \tilde{m}), & \text{if } a \geq \bar{a}(m, \tilde{m}) \\ a, & \text{if } a < \bar{a}(m, \tilde{m}) \end{cases} \tag{13}$$

$$p(m, \tilde{m}, a) = \frac{1}{\varphi} \tag{14}$$

Proof. See the appendix. □

The "cutoff" level $\bar{a}$ captures the amount of assets that type 1 needs to sell in order to acquire the case-specific optimal monetary transfer. Why is that transfer 'case-specific'? Because it depends on the money holdings of the two parties: if $m + \tilde{m} \geq m^*$, the money of the two agents pulled together allows the type 1 buyer to purchase q^* in the DM. In this case, the optimal (real) money transfer is $\varphi(m^* - m)$, i.e., the type 1 wants to acquire the amount of money that she is missing in order to afford q^*. If, on the other hand, $m + \tilde{m} < m^*$, the type 1 buyer will not be able to purchase q^*, even if she acquired all of the type 2's money. In this case, the optimal (real) monetary transfer is $\varphi\tilde{m}$, i.e., the type 1 buyer should acquire all the type 2's money.

Having defined the case-specific optimal money transfer, the remaining question is "Can the type 1 buyer afford it?" The answer depends on whether her asset holdings, a, are enough to cover the cutoff levels $\bar{a}$ (which, clearly, are also case-specific). If yes, then the type 1 will give away exactly $\bar{a}$ units of assets and obtain the optimal amount of money. If not, she will give away all of her assets and obtain a less-than-optimal amount of money. Notice that the OTC asset price is always equal to $1/\varphi$, i.e., $p\varphi = 1$. Since this asset is about to yield one unit of numeraire, the last expression simply says that the type 2 agent cannot purchase assets at a discount but she must pay the full price, due to the fact that she has no bargaining power.

3.4 Optimal Choices

As is standard in models that build on Lagos and Wright (2005), the representative buyer's portfolio decision does not depend on her trading history. Simply put, all buyers in the CM will choose the same portfolio $(\hat{m}, \hat{a})$ for the next period, regardless of their type, whether they matched in the OTC market, etc. This optimal decision is described by maximizing the agent's objective function, call it J, which can be derived as follows. First, substitute (4) and (5) into (3) to obtain an expression for $\mathbb{E}[\Omega(m, a)]$. Then, substitute that expression into (1), exploiting (2) and (6), and focus only on the terms that contain the portfolio choices $(\hat{m}, \hat{a})$ inside the maximum operator (the rest do not affect the agent's optimal choice). This yields:

$$\begin{aligned} J(\hat{m}, \hat{a}) \equiv &-\varphi\hat{m} - \psi\hat{a} \\ &+ \beta\big\{(1-\lambda)\big[\alpha\lambda\big(u(q_1(\hat{m}+\chi/\hat{\varphi})) + \hat{a} - \chi\big) + (1-\alpha\lambda)\big(u(q_1(\hat{m})) + \hat{a}\big)\big] \\ &+ \lambda\big[\alpha(1-\lambda)\big(u(q_2(\hat{m}-\tilde{\chi}/\hat{\varphi}, \hat{a}+\tilde{\chi})) + \hat{\varphi}(\hat{m}-\tilde{\chi}/\hat{\varphi}) + \hat{a} + \tilde{\chi} - \rho(\hat{m}-\tilde{\chi}/\hat{\varphi}, \hat{a}+\tilde{\chi})\big) \\ &+ [1-\alpha(1-\lambda)]\big(u(q_2(\hat{m}, \hat{a})) + \hat{\varphi}\hat{m} + \hat{a} - \rho(\hat{m}, \hat{a})\big)\big]\big\}. \end{aligned} \quad (15)$$

Naturally, the first line of J represents the cost of purchasing the portfolio $(\hat{m}, \hat{a})$, and the remaining lines represent the expected benefit from carrying that portfolio into the next period. For instance, consider the second line, which captures the benefit from being a type 1 buyer (an event that takes place with probability $1-\lambda$). That buyer may be able to trade in the OTC market (with probability $\alpha\lambda$), in which case she can sell χ units of assets and boost her money holdings by $\chi/\hat{\varphi}$. If the buyer does not match in the OTC (with probability $1-\alpha\lambda$), she will simply continue to the DM with her original money holdings. The third and fourth lines, which capture the benefit from being a type 2 buyer, admit a similar interpretation.[13]

Before we proceed to a formal analysis of the buyer's optimal portfolio choice, it is useful to provide some intuition. For any given price, $\hat{\varphi}$, and belief about the portfolio that other agents carry, $(\tilde{m}, \tilde{a})$, the representative buyer realizes that her own portfolio choice will bring her in a different "branch" of the OTC bargaining solution (see Lemma 2). To determine these branches, three questions are relevant. First, if I turn out to be type 1, will my money together with the type 2's money suffice to purchase q^* in the DM (is $\hat{m} + \tilde{m} \geq m^*$)? Second, if I turn out to be a type 1, are my assets enough to acquire the (case-specific) optimal amount of money from the type 2 (is $\hat{a} \geq \bar{a}$)? Third, if I turn out to be a type 2 buyer, are my own liquid assets enough to purchase q^* in the DM (is $\hat{\varphi}\hat{m} + \hat{a} \geq z(q^*)$)? It turns out that there are four relevant regions:[14]

[13] By Lemma 2, the amount of assets that changes hands in the OTC depends on the money and asset holdings of type 1, and (only) on the money holdings of type 2. Consequently, the amount of assets traded in the OTC when the representative buyer is type 1, χ, will typically be different than the amount traded when the representative buyer is type 2, $\tilde{\chi}$. Thus, if the representative buyer holds the portfolio $(\hat{m}, \hat{a})$ and expects her typical trading partner (in the OTC) to hold the portfolio $(\tilde{m}, \tilde{a})$, then χ depends on the terms $(\hat{m}, \tilde{m}, \hat{a})$, but $\tilde{\chi}$ depends on the terms $(\tilde{m}, \hat{m}, \tilde{a})$.

[14] The buyer will never bring more money than what is necessary to afford q^*. Thus, in what follows, $\hat{m} \leq m^*$.

Region 1: $\hat{m} + \tilde{m} \geq m^*$, $\hat{a} \geq \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} \geq z(q^*)$. There is enough money in the OTC match to allow the type 1 to purchase q^* in the DM. If the agent is of type 1, her asset holdings allow her to acquire the critical amount of money $m^* - \hat{m}$. If the buyer is of a type 2, her total liquid assets allow her to purchase q^* in the DM.

Region 2: $\hat{m} + \tilde{m} \lesseqgtr m^*$, $\hat{a} < \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$. If the buyer is of type 1, her asset holdings are not enough to acquire the optimal amount of money from the type 2. If she is of type 2, her total liquid balances are not enough to purchase q^* in the DM. (Whether $\hat{m} + \tilde{m}$ exceeds m^* or not is irrelevant because the asset holdings are scarce anyway).

Region 3: $\hat{m} + \tilde{m} < m^*$, $\hat{a} \geq \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$. The total money in the OTC meeting is not enough to allow the type 1 buyer to purchase q^* in the DM. If the buyer is of type 1, her asset holdings are enough to acquire all the all the money of the type 2 buyer. If the buyer is of type 2, her total liquid balances are not enough to purchase q^* in the DM.

Region 4: $\hat{m} + \tilde{m} < m^*$, $\hat{a} \geq \bar{a}$, $\hat{\varphi}\hat{m} + \hat{a} \geq z(q^*)$. The total money in the OTC meeting is not enough to allow the type 1 buyer to purchase q^* in the DM. If the buyer is of type 1, her asset holdings are enough to acquire all the money of the type 2 buyer. If the buyer is of type 2, her total liquid balances are enough to purchase q^* in the DM.

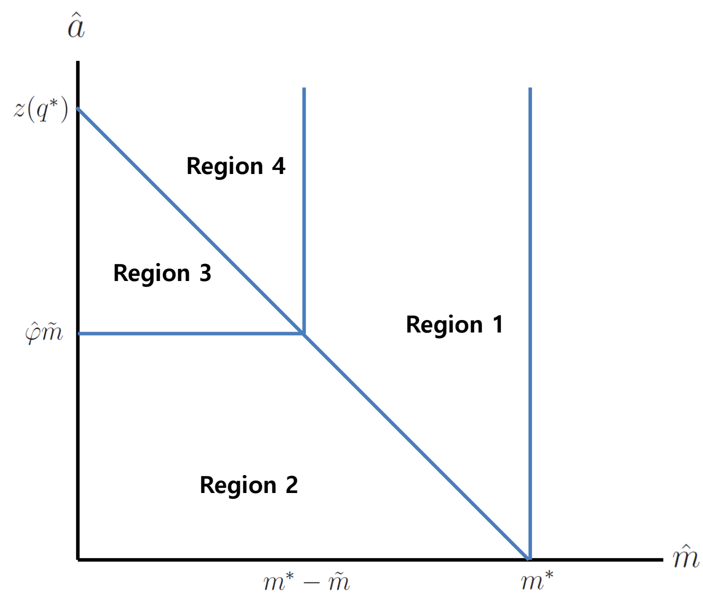


Figure 2: Regions of the individual optimization problem, in terms of money holdings.

Figure 2 illustrates the four regions. Why are they relevant? Because the region where the buyer finds herself in, is crucial for determining the benefit of the marginal unit of money/assets, which, of course, is crucial for determining the demand functions. Let us illustrate this through some examples. Suppose that given the price, $\hat{\varphi}$, and beliefs, $(\tilde{m}, \tilde{a})$, the representative buyer contemplates a portfolio choice that brings her in Region 1. Within that region, carrying an additional unit of the asset has no direct liquidity benefit (if I am a type 2 buyer I can already purchase q^*) or indirect liquidity benefit (if I am a type 1 buyer I can already acquire in the OTC the money I am missing in order to get to q^*). Hence, in that region, the buyer values an additional unit of asset only for its dividend, not for its liquidity. Does the buyer value an additional unit of money for its liquidity? Yes, because that unit helps her buy additional goods in the event of being a type 1 buyer who did not match in the OTC. As another example consider Region 2. Here, the marginal unit of real assets is valued both for its direct and indirect liquidity: direct, because $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$, so an additional unit of assets can help a type 2 buyer increase DM consumption, and indirect, because $\hat{a} < \bar{a}$, so an additional unit of assets can help a type 1 buyer acquire more money in the OTC.

We now provide a formal description of the representative buyer's optimal choice.

Lemma 3. *The function $J : \mathbb{R}^2 \to \mathbb{R}$ has the following properties:*

i. It is continuous and differentiable within all the regions.

ii. It is strictly concave in $\hat{m}$ and weakly concave in $\hat{a}$.

iii. It is weakly concave in its whole domain.

Let $J_j^i(\hat{m}, \hat{a})$, $j = 1, 2$, stands for the derivative of the objective function in Region $i = 1, 2, 3, 4$ with respect to the $j - th$ argument. Then, we have:

$$J_1^1(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\sigma_1 + (1-\sigma_1)L(\hat{\varphi}\hat{m})\big] + \lambda\Big\}, \tag{16}$$

$$J_2^1(\hat{m}, \hat{a}) = J_2^4(\hat{m}, \hat{a}) = -\psi + \beta, \tag{17}$$

$$J_1^2(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\sigma_1 L(\hat{\varphi}\hat{m} + \hat{a}) + (1-\sigma_1)L(\hat{\varphi}\hat{m})\big] + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{18}$$

$$J_2^2(\hat{m}, \hat{a}) = -\psi + \beta\Big\{(1-\lambda)\big[\sigma_1 L(\hat{\varphi}\hat{m} + \hat{a}) + (1-\sigma_1)\big] + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{19}$$

$$J_1^3(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\sigma_1 L(\hat{\varphi}(\hat{m} + \tilde{m})) + (1-\sigma_1)L(\hat{\varphi}\hat{m})\big] + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{20}$$

$$J_2^3(\hat{m}, \hat{a}) = -\psi + \beta\Big\{(1-\lambda) + \lambda L(\hat{\varphi}\hat{m} + \hat{a})\Big\}, \tag{21}$$

$$J_1^4(\hat{m}, \hat{a}) = -\varphi + \beta\hat{\varphi}\Big\{(1-\lambda)\big[\sigma_1 L(\hat{\varphi}(\hat{m} + \tilde{m})) + (1-\sigma_1)L(\hat{\varphi}\hat{m})\big] + \lambda\Big\}, \tag{22}$$

where we have defined $L(\cdot) \equiv (h \circ z^{-1})(\cdot)$, with $h(q_i) \equiv \frac{u'(q_i)}{z'(q_i)}$. The liquidity premium function L satisfies $L(\cdot) \geq 1$ and $L'(\cdot) < 0$.

Proof. See the appendix. □

Lemma 4. *Taking prices, $(\varphi, \hat{\varphi}, \psi)$, and beliefs, $(\tilde{m}, \tilde{a})$, as given, the optimal portfolio choice of the representative buyer, $(\hat{m}, \hat{a})$, can be characterized as follows:*

i. It satisfies $J^i_j(\hat{m}, \hat{a}) = 0$, *for all* i, j.

ii. If $\varphi > \beta\hat{\varphi}$ *and* $\psi = \beta$, *there exists a unique* $\hat{m}$, *whereas any* $\hat{a}$ *is optimal as long as* $(\hat{m}, \hat{a})$ *is in Regions 1, 4, or on the boundary between them.*

iii. If $\varphi > \beta\hat{\varphi}$ *and* $\psi > \beta$, *there exists a unique optimal portfolio choice* $(\hat{m}, \hat{a})$, *which lies in Regions 2, 3, or on the boundary between them.*

Proof. See the appendix. □

Naturally, the optimal portfolio choice of the buyer amounts to equating the marginal cost of each asset (φ for money and ψ for the real asset) to its marginal benefit, which depends on the relevant region. If $\psi = \beta$, the net cost of carrying assets across periods is zero, thus, optimality dictates that the buyer bring an amount of assets high enough to exploit *all* possible liquidity properties (direct and indirect), and this can only happen in Regions 1 and 4. The buyer is only willing to buy the asset at a price higher than the fundamental value, i.e., $\psi > \beta$, if the marginal unit is still helpful for liquidity purposes, which is true only in Regions 2 and 3.

Of course, the asset's direct and indirect liquidity properties affect not only its own demand (and price), but also the demand for money. While interesting, examining the money demand is not of first-order importance for the analysis, so we relegate it to Appendix A.1.

3.5 Equilibrium

With the optimal behavior of the representative buyer laid out, it is now straightforward to characterize equilibrium, and we focus on symmetric, steady state equilibria, where $\varphi M = \hat{\varphi}\hat{M}$, implying that $\varphi/\hat{\varphi} = 1 + \mu$.

Definition 1. *A steady state equilibrium is a list* $\{\psi, \chi, \mathrm{w}_1, \mathrm{w}_2, q_1^n, q_1^m, q_2^n, q_2^m\}$. *The terms* ψ, χ *have already been defined. The remaining equilibrium objects are as follows:* $\mathrm{w}_1 = \varphi M$ *and* $\mathrm{w}_2 = \varphi M + A$, *represent the real liquid balances in a type 1 and a type 2 DM meeting, respectively;* $q_1^n(q_2^n)$ *stands for the amount of DM good traded in a type 1 (type 2) DM meeting, when the buyer was* ***not matched*** *in the preceding OTC market;* $q_1^m(q_2^m)$ *is the analogous expression for the case in which the buyer was* ***matched*** *in the OTC. The equilibrium objects are such that:*

i. Given prices, the representative buyer's portfolio choice maximizes her objective function, i.e., it satisfies Lemma 4.

ii. The equilibrium quantity q_1^n *is given by* $q_1^n = z^{-1}(\mathrm{w}_1)$. *The quantities* q_1^m, q_2^n, *and* q_2^m *can be obtained as follows:*

$$q_1^m = \begin{cases} q^*, & \text{in Region 1} \\ z^{-1}(\mathrm{w}_2), & \text{in Region 2} \\ z^{-1}(2\mathrm{w}_1), & \text{in Region 3 and 4} \end{cases}$$

$$q_2^n = q_2^m = q_2 \quad = \quad \begin{cases} q^*, & \text{in Region 1 and 4} \\ z^{-1}(\mathrm{w}_2), & \text{in Region 2 and 3} \end{cases}$$

iii. *The amount of assets traded in the OTC, χ, satisfies (13).*

iv. *Each market clears and expectations are rational: $\hat{m} = \tilde{m} = (1+\mu)M$, and $\hat{a} = \tilde{a} = A$.*

The amount of good traded in the DM depends on the type of the meeting (which depends on whether the seller accepts assets or not) and, in the case of a type 1 meeting, on whether the buyer was matched in the preceding OTC market. If the type 1 buyer did not match, the amount of q she can purchase depends only on her own real balances, i.e., $q_1^n = z^{-1}(\mathrm{w}_1)$. If she did match, her post-OTC trade money balances depend on the specific region. In Region 1, both money and assets are plentiful, hence, the type 1 buyer will obtain q^*. In Regions 3, 4, assets are plentiful, but money is not. Hence, the type 1 buyer will acquire all the money of the type 2 (by symmetry, this implies that she will enter the DM with real balances equal to $2\mathrm{w}_1$), and purchase the quantity $q_1^m = z^{-1}(2\mathrm{w}_1) < q^*$. In Region 2, the type 1 buyer's assets do not allow her to purchase the optimal amount of money from the type 2. Thus, here, the amount of DM good purchased by type 1 also depends on A, specifically, $q_1^m = z^{-1}(\mathrm{w}_2) < q^*$, where $\mathrm{w}_2 = \mathrm{w}_1 + A$. Notice that the amount of DM good purchased by a type 2 buyer, q_2, is irrelevant of whether she matched in the OTC, because that type has no bargaining power in the OTC market.

Lemma 5. *A steady state equilibrium exists and is unique.*

Proof. See the appendix. □

3.6 Characterization of Equilibrium

Having defined equilibrium and established its existence and uniqueness, we now proceed to the characterization of the key equilibrium variables, namely, the asset price ψ and the DM production, q, for an exogenous probability of asset acceptance by sellers, λ. To that end, it useful to understand how the various regions of equilibrium look in the aggregate economy (Figure 2 illustrated the various regions from the perspective of the representative agent). Figure 3 does precisely that, i.e., it illustrates the four regions of equilibrium, not as functions of the individual choices $\hat{a}$ and $\hat{m}$, but as functions of the exogenous asset supply A (which, in equilibrium, is equal to $\hat{a}$), and the policy parameter μ (which, in equilibrium, is the main driver of $\hat{m}$).

While the details of the derivation of Figure 3 are relegated to Appendix A.2, the intuition is straightforward. Region 1 represents the region where not only type 2 but also matched type 1 buyers are able to attain q^* in the DM. Naturally, this happens when A is relatively high and μ, which in steady state is the inflation rate, is relatively low. Now, suppose that the asset supply

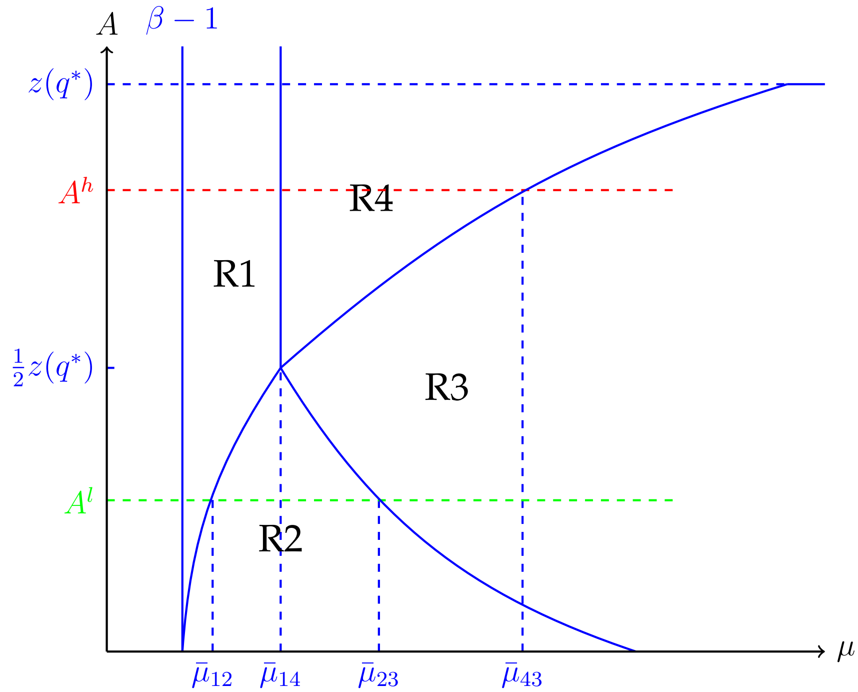


Figure 3: Aggregate regions of equilibrium, in terms of money growth and asset supply.

is relatively high, say $A = A^h$, as in the figure, and consider an increase in μ keeping A constant. As μ increases, the equilibrium real balances w_1 decrease, and soon the matched type 1 buyer will not be able to acquire the amount of money that would allow her to purchase q^* (although a type 2 buyer can still afford q^*). In a sense, the type 1's asset is plentiful, but the aggregate amount of money in the OTC is not. This is precisely what is going on in Region 4. If μ kept increasing, then the real balances w_1 would decrease so much that, eventually, even the type 2 buyer would not afford q^*. In other words, we would now be in Region 3.

What if the asset supply was relatively low, say $A = A^l$, as in the figure? First notice that even for such low asset supply, we can still be in Region 1, but this would require an extremely low μ. As μ increases, the equilibrium real balances w_1 decrease. With A so low, and with inflation on the rise, but still in intermediate levels, we are in a region where type 2 buyers cannot afford q^*, and matched type 1 buyers do not have enough assets to acquire the amount of money they would wish in the OTC. As μ increases further, and with A fixed, an interesting development takes place: assets are still not enough to allow type 2 buyers to purchase q^*, but they are enough to allow type 1 buyers to acquire all the real money balances of type 2 buyers in the OTC, because these balances are now very little; in other words, we are now in Region 3.

The following proposition formalizes the results concerning equilibrium asset prices.

Proposition 1. *Let $\bar{\mu}_{ij}$, $i, j \in \{1, 2, 3, 4\}$, denote the values of μ that determine the boundary points between Regions i and j, for any given asset supply A; these boundaries are defined in Lemma 8 in the appendix. Then, equilibrium asset prices are as follows:*

Case 1: If $A \geq z(q^*)$, then, for any $\mu > \beta - 1$, we have $\psi = \beta$;

Case 2: If $A \in (z(q^*)/2, z(q^*))$, then,

i. For all $\mu \in (\beta - 1, \bar{\mu}_{43}]$, we have $\psi = \beta$;

ii. For all $\mu > \bar{\mu}_{43}$, the CM asset price exceeds the fundamental value, and it is a strictly increasing function of μ, i.e., $\psi = \psi(\mu) > \beta$, and $\psi'(\mu) > 0$;

Case 3: If $A < z(q^*)/2$, then

i. For all $\mu \in (\beta - 1, \bar{\mu}_{12}]$, we have $\psi = \beta$;

ii. For all $\mu > \bar{\mu}_{12}$, the CM asset price exceeds the fundamental value, and it is a strictly increasing function of μ, i.e., $\psi = \psi(\mu) > \beta$, and $\psi'(\mu) > 0$.

Proof. See the appendix. □

The key observation is that agents will be willing to pay liquidity premia, only if the marginal unit of the asset is still useful for (direct or indirect) liquidity properties. Thus, for any $A \in (z(q^*)/2, z(q^*))$, there will exist $\bar{\mu}_{43}$ such that $\mu \leq \bar{\mu}_{43}$ will bring us in Region 1 or 4. Within either one of these regions, all the liquidity properties of the asset have been exploited, so we must have $\psi = \beta$. The same is true if $A < z(q^*)/2$ and $\mu \in (\beta - 1, \bar{\mu}_{12}]$, since these parameter values imply an equilibrium within Region 1. If $A < z(q^*)/2$ and $\mu \in (\bar{\mu}_{12}, \bar{\mu}_{23})$, equilibrium lies within Region 2, where the marginal unit of the asset serves both direct and indirect liquidity properties, and this will be reflected in the price. As we move from Region 2 to Region 3, say, because μ increases beyond $\bar{\mu}_{23}$, for some given $A < z(q^*)/2$, the marginal unit of the asset is still providing direct liquidity services, but not indirect.

These results are depicted in Figure 4, where ψ is depicted as a function of μ for two levels of asset supply: $A^h \in (z(q^*)/2, z(q^*))$ and $A^l < z(q^*)/2$. Notice that within the regions where the marginal asset is valued for its liquidity (and, hence, $\psi > \beta$), we also have $\psi'(\mu) > 0$: a higher inflation depresses equilibrium real balances and makes the asset more valuable for its liquidity, regardless of whether this liquidity is direct (as in Regions 2,3) or indirect (as in Region 2). Also, notice that within Region 3 the slope of ψ with respect to μ is the same, regardless of whether $A = A^h$ or $A = A^l$, but, naturally, the equilibrium price is higher under $A = A^l$ because, with a low asset supply, the marginal valuation of agents for the liquidity properties of the asset is higher. The slope of ψ is the highest within Region 2 (which is only relevant if $A = A^l$), because this is precisely where both direct and indirect liquidity kick in.

The next proposition summarizes the results concerning equilibrium trade in the DM.

Proposition 2. *Let Q denote the average trade volume in the DM, given by $Q = (1-\lambda)[\alpha\lambda q_1^m + (1-\alpha\lambda)q_1^n] + \lambda q_2$. Then, Q is a strictly decreasing function of μ for any $A \leq z(q^*)$.*

Proof. See the appendix. □

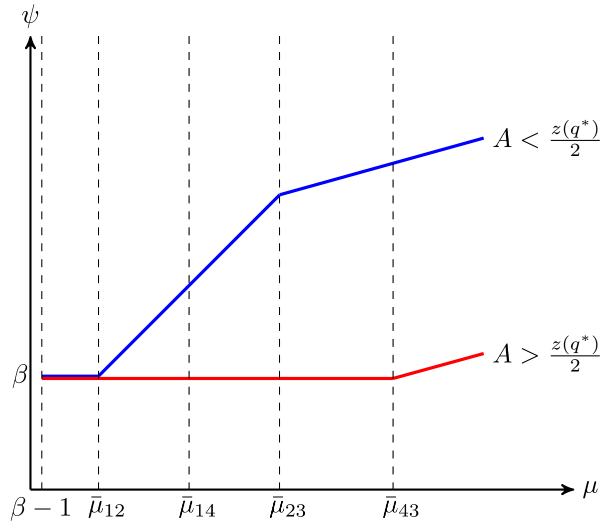


Figure 4: Effects of money growth on the asset price

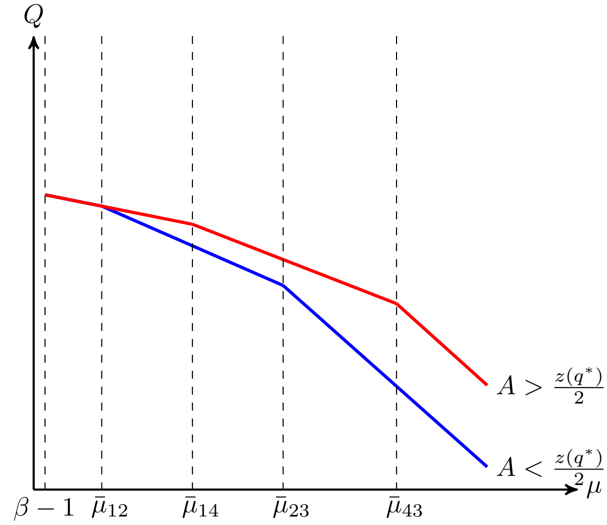


Figure 5: Effects of money growth on the trade volumes in the DM

Proposition 2 is depicted in Figure 5, where the average DM production, Q, is plotted against μ, for $A^h \in (z(q^*)/2, z(q^*))$ and $A^l < z(q^*)/2$. Notice that Q is unaffected by asset supply only within Region 1 (i.e., for $\mu \leq \bar{\mu}_{12}$), because in that case both type 2 and matched type 1 buyers can afford q^*. However, even within that region, Q is decreasing in μ, because $\partial \mathrm{w}_1/\partial \mu < 0$, and the quantity purchased by unmatched type 1 buyers depends positively on w_1.[15] For any $\mu > \bar{\mu}_{12}$, the average DM production is higher under $A = A^h$, because a higher asset supply allows type 2 buyers to purchase more goods in the DM, and matched type 1 buyers to acquire more money in the OTC market.

One may also ask how the OTC trade volume looks as a function of λ (which until now has been exogenous). It turns out that it is hump-shaped. This result is intuitive once we consider the two extremes, i.e., $\lambda = 1$ or 0. When all sellers accept assets as MOE, the role of trade in the OTC market vanishes, and the volume of trade is zero. On the other extreme, when no sellers accept assets, every agent would like to sell assets for money in the OTC, but no one is willing to supply it, because money is the exclusive medium of exchange for all agents in this economy. The end result is the same: no trade in the OTC market. Thus, the volume of trade in the OTC is zero at the extremes, and positive for intermediate values of λ.

[15] Formally, $q_1^n = z^{-1}(\mathrm{w}_1)$, where $z^{-1}(.)$ is a strictly increasing function.

4 Endogenous Asset Acceptance

4.1 Optimal Choice of the Seller

Having analyzed equilibrium for any exogenous λ, the task of this section is to determine this important term endogenously. Following LPW, we assume that sellers have the option to pay (*ex ante*) an information cost κ, which allows them to recognize the quality of real assets and, consequently, accept them as means of payment. Clearly, a seller's profit depends on her own choice to invest in the technology that allows her to accept assets, and the decision of other sellers to acquire that technology. Let λ denote the representative seller's belief about the measure of (other) sellers that accept assets as MOE. Following a standard method in monetary theory, first introduced by Kiyotaki and Wright (1989), we will construct symmetric equilibria where the best response function of the representative seller intersects with the 45 degree line.[16]

The *net* profit of a seller who acquires the information, for some $\lambda \in [0, 1]$, is:

$$\Pi(\lambda) \equiv \beta(1-\eta)\left\{\underbrace{[u(q_2(\lambda)) - q_2(\lambda)]}_{\text{Type 2 Profit}} - \underbrace{\alpha\lambda[u(q_1^m(\lambda)) - q_1^m(\lambda)]}_{\text{Matched Type 1 Profit}} - \underbrace{[1-\alpha\lambda]\,[u(q_1^n(\lambda)) - q_1^n(\lambda)]}_{\text{Not Matched Type 1 Profit}}\right\},$$

where q_2 stands for the DM good traded in a type 2 meeting, and q_1^m (q_1^n) stands for the DM good traded in a type 1 meeting, if the buyer matched (did not match) in the OTC. The interpretation of Π is straightforward. With proportional bargaining, the seller always earns a fraction $1-\eta$ of the total DM surplus. What is that surplus? If the seller pays κ, she is, by definition, a type 2 seller, and her DM transaction will generate a surplus equal to $u(q_2) - q_2$. However, by paying that cost she gives up the surplus that would have been generated in a type 1 meeting, and which depends on whether the buyer was matched in the OTC (with probability $\alpha\lambda$) or not (with probability $1-\alpha\lambda$). Letting $\Lambda(\lambda)$ denote the seller's optimal response to her belief λ, it is clear that she will choose $\Lambda = 1$, if and only if $\Pi(\lambda) > \kappa$.

Inspection of the definition of Π, reveals that λ affects a seller's profit through two channels. First, it directly affects the probability with which the buyer in a type 1 meeting was matched in the OTC. Second, it indirectly affects the DM good traded in the various contingencies, because λ is an important determinant of the demand for the various assets. Studying the properties of

[16] Following that paper, we interpret λ either as the *measure* of sellers who accept assets or as the *probability* with which other sellers accept assets. For any given λ, the representative seller will choose her own probability of accepting assets, say Λ, so that her profit is maximized. Any point where this "best response" function intersects with the 45 degree line is automatically a symmetric equilibrium because it implies optimal behavior (it belongs to the seller's *best response* function) and symmetry (it satisfies $\Lambda = \lambda$).

$\Pi(\lambda)$ is our main goal in the remainder of Section 4.1. We start with an auxiliary lemma.

Lemma 6. *In the steady state equilibrium, and in any possible region, we have* $d\mathrm{w}_1/d\lambda \leq 0$ *and* $d\mathrm{w}_1/d\alpha \leq 0$.

Proof. See the appendix. $\square$

The lemma states that the asset acceptance rate, λ, and the OTC matching efficiency, α, effectively act as inflation: a higher λ induces buyers to carry fewer money balances, since they expect to be able to use assets as MOE with higher probability. Similarly, a higher α induces buyers to carry less money, since they expect that it will be easier to get extra cash in the OTC, if they need it. Consequently, Lemma 6 reveals a complication in our task to characterize the shape of $\Pi(\lambda)$: if changes in λ mimic changes in inflation (or μ), the seller who chooses her best response $\Lambda(\lambda)$, must take under consideration that different λ's may be associated with different equilibrium regions; and we now know that the various equilibrium q's are very different in each region. To see this point, suppose that (A, μ) are indicated by the red dot in Figure 6 (and satisfy $A \in (z(q^*)/2, z(q^*))$ and $\mu < \bar{\mu}_{14}$). Given these parameters, $\lambda = 0$ would imply an equilibrium in Region 1 (upper-left panel). However, as λ increases, the boundaries of the various regions start moving westward, and, eventually, there comes a point where the red dot lies within Region 4 (upper-right panel). Thus, equilibrium now lies in a different region, even though (A, μ) *did not change*. As λ increases further, equilibrium will eventually lie in (the even scarcer) Region 3 (lower-left panel). Clearly, different parameters would lead to different "paths".[17,18]

With this discussion in mind, we are now ready to study the properties of $\Pi(\lambda)$ and, consecutively, the equilibrium with endogenous λ.

Proposition 3. *a) The derivative of* Π *with respect to* λ *is given by:*

$$\begin{aligned}\Pi'(\lambda) =& \beta[1-\eta]\left\{\frac{\partial q_2}{\partial \lambda}[u'(q_2(\lambda))-1]\right.\\ &-\alpha\Big[[u(q_1^m(\lambda))-q_1^m(\lambda)]-[u(q_1^n(\lambda))-q_1^n(\lambda)]\Big]\\ &\left.-\alpha\lambda\frac{\partial q_1^m}{\partial \lambda}[u'(q_1^m(\lambda))-1]-(1-\alpha\lambda)\frac{\partial q_1^n}{\partial \lambda}[u'(q_1^n(\lambda))-1]\right\}.\end{aligned}$$

[17] For instance, if $A < z(q^*)/2$ and $\mu < \bar{\mu}_{12}$, the red dot would lie in the southwest portion of Region 1 (for $\lambda = 0$), and an increase of λ from 0 to 1 would have brought us through Regions 1, 2, and 3, consecutively. Or, if $A \in (z(q^*)/2, z(q^*))$ and $\mu > \bar{\mu}_{43}$, then equilibrium would lie in Region 3 for any $\lambda \in [0, 1]$.

[18] As λ increases, some regions of equilibrium vanish. As we can see in the lower-right panel of Figure 6, the first region to vanish is Region 1. This is intuitive: Region 1 is the region where all types of buyers (except unmatched type 1) get the first best–it is the region of plentifulness. An increase in λ depresses real money balances and makes it impossible for some types to attain q^*. If λ increases further, it is not a surprise that the second region to disappear is the "second most plentiful" region, i.e., Region 4 (that is the region where matched type 1 buyers do not attain q^*, but type 2 buyers do). This is precisely what we see in the lower-right panel of the figure, where the only regions left are 2 and 3. Notice that this plot assumes $\lambda = 1$, but it would look identical for λ in a neighborhood of 1.

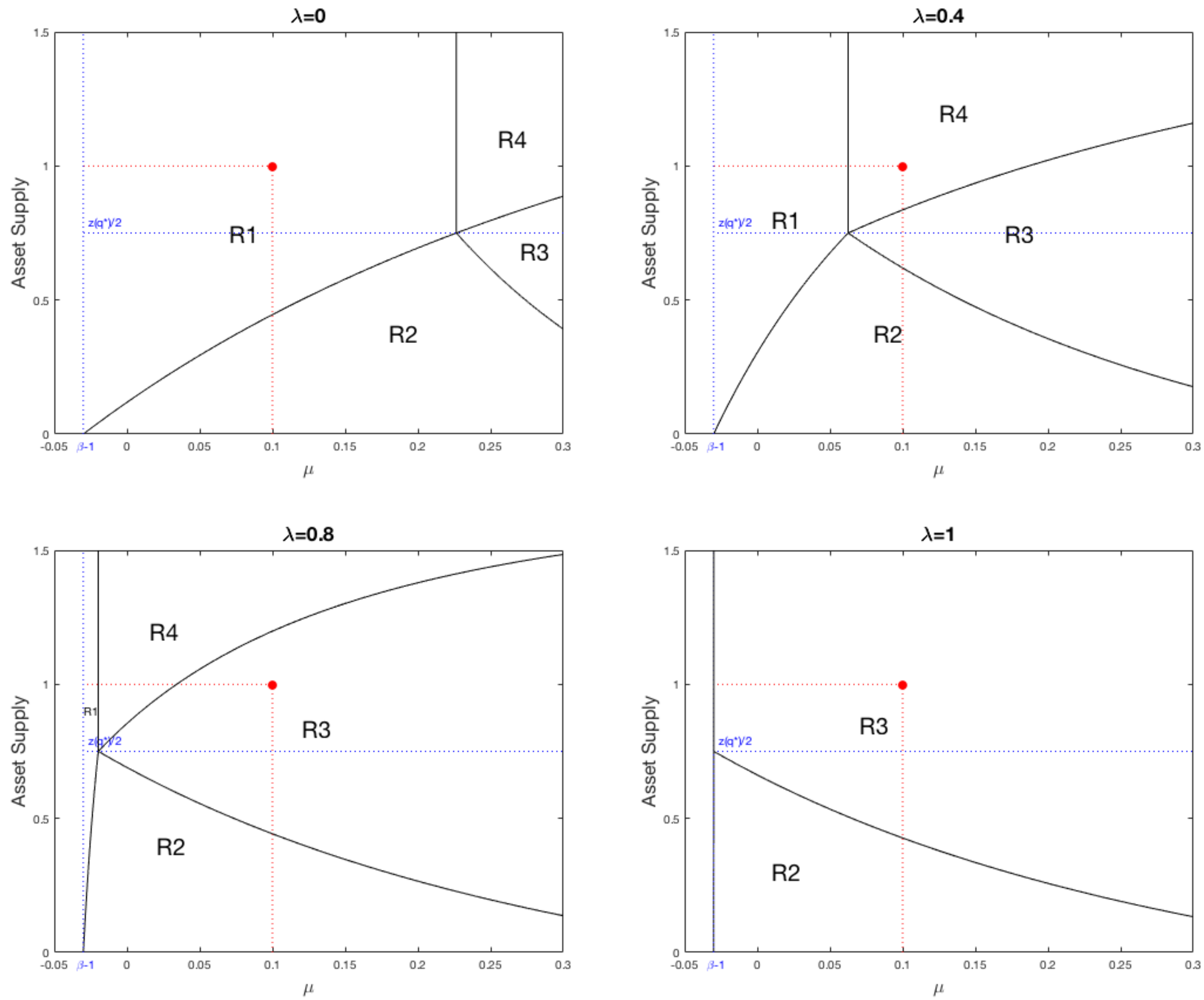


Figure 6: Aggregate regions of equilibrium with different levels of λ. All four cases assume $u(x) = x^{1-\rho}/(1-\rho)$, $\beta = 0.97$, $\rho = 0.5$, $\eta = 0.5$, and $\alpha = 1$.

b) In the steady state equilibrium, a sufficient condition for $\Pi'(\lambda) > 0$ in Region 1 is that u' is log-concave, i.e., $(u'')^2 > u'u''$. In all other regions, the sign of $\Pi'(\lambda)$ is ambiguous. Furthermore, $\partial\Pi/\partial\alpha > 0$ in Region 1 for any parameter values. In all other regions, the sign of $\partial\Pi/\partial\alpha$ is ambiguous.

Proof. See the appendix. □

As we have already discussed, changes in λ affect Π through two channels. Proposition 3 reveals that these channels have opposite directions. On the one hand, a high λ induces buyers to carry few real money balances, as they expect to be able to use their assets as MOE; this channel tends to make Π increasing in λ because a seller who chooses to not get informed has a lot to lose. On the other hand, a high λ implies a high probability of matching for type 1 buyers in the OTC; a seller who did not get informed is very likely to meet a buyer who got matched in the OTC and was, therefore, able to boost her money holdings. Thus, the loss from not acquiring information is not that significant.

Although Section 4.2 provides a detailed comparison of our model with that of LPW, a quick point of comparison is in order here. The first of these two channels is also present in LPW. In fact, it is the main driver of Π, which is why in that paper Π is strictly increasing in the measure of informed sellers. However, the second channel, which tends to make Π decreasing in λ, is unique to our model, making the analysis more complicated, but also interesting.

A few more details about Proposition 3 are worth emphasizing. While, the two opposing forces make it difficult to pin down the sign of $\Pi'(\lambda)$ for all parameter values, we are able to show (under slightly stronger assumptions) that $\Pi'(\lambda) > 0$ in Region 1.[19] Why is $\Pi(\lambda)$ increasing in Region 1, but not necessarily so in other regions? In Region 1, the first of the two aforementioned channels is quantitatively important, because type 2 buyers can attain q^*, and so type 2 meetings in the DM produce the maximum surplus possible, $u(q^*) - q^*$. Thus, a seller who chooses to not get informed loses a lot. Of course, a seller who does not pay κ will meet with a type 1 buyer, and that buyer's probability of matching in the OTC is increasing in λ. But as discussed earlier in this section, being in Region 1 means that λ is likely to be small anyway. Thus, the relative benefit from not acquiring information is not quantitatively significant. As λ increases, and we move into regions where liquidity is more scarce, the first (positive) force weakens, and the second (negative) force becomes stronger, leading to an ambiguous effect on Π.

4.2 Characterization of Endogenous λ

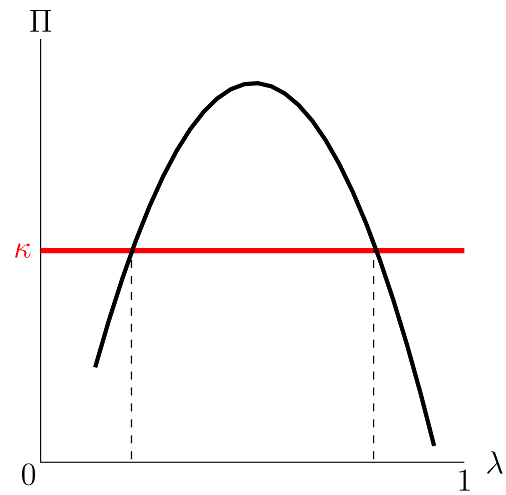
The following lemma describes the equilibrium value of λ, which, as already discussed, will be any point where the representative seller's best response function intersects with the 45^o line.

Lemma 7. *The equilibrium value of the λ is as follows:*

1. *When $\Pi(\lambda) < \kappa$, $\forall \lambda \in [0, 1]$, there exists a unique pure strategy Nash equilibrium with $\lambda = 0$.*
2. *When $\Pi(\lambda) > \kappa$, $\forall \lambda \in [0, 1]$, there exists a unique pure strategy Nash equilibrium with $\lambda = 1$.*
3. *If there exist values $\lambda \in [0, 1]$, such that $\Pi(\lambda) = \kappa$, then these values constitute mixed strategy Nash equilibria.*
4. *If multiple mixed strategy Nash equilibria, λ, exist, only those that satisfy $\Pi'(\lambda) < 0$ are stable.*

Proof. The proof is obvious, hence, omitted. □

[19] In fact, under a wide range of numerical simulations, we were not able to find parameter values for which $\Pi'(\lambda) < 0$ in Region 1. However, for an analytical solution, we need to impose slightly stronger assumptions on u. For details, see the proof of Proposition 3 in the appendix.



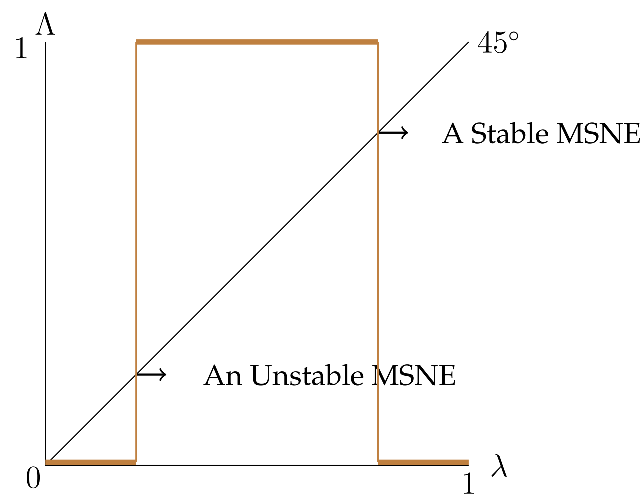


Figure 7: Mixed strategy Nash equilibria for λ when $\Pi(\lambda)$ is hump shaped

The interpretation of Lemma 7 is straightforward. An individual seller optimally chooses to become type 2, if $\Pi(\lambda) > \kappa$ for all λ; if this is the case, the unique symmetric equilibrium involves all sellers investing in the technology, i.e., $\lambda = 1$, and the real asset becomes a perfect substitute to money. Clearly, this equilibrium is likely to arise when κ is very small. On the other extreme, if $\Pi(\lambda) < \kappa$ for all λ, the unique equilibrium has no sellers investing, i.e., $\lambda = 0$, and the real asset is fully illiquid. If there exist values of λ for which $\Pi(\lambda) = \kappa$, any $\Lambda \in [0, 1]$ is a best response, hence, such λ's constitute mixed strategy Nash equilibria. We find these "interior" equilibria particularly interesting because they imply that assets are partially liquid, which is the most empirically relevant case. (LPW make a similar argument.)

However, part (4) of Lemma 7 provides a word of caution: only interior equilibria with $\Pi'(\lambda) < 0$ are stable: if an arbitrarily small measure ε of sellers accidentally accept assets, the representative seller's best response is to not accept (i.e., she does not have an incentive to follow the deviant sellers). In contrast, if $\Pi'(\lambda) > 0$, and ε sellers accidentally accepted assets, the individual seller would have an incentive to follow them, thus, "unstabilizing" the equilibrium. Figure 7 illustrates the determination of equilibrium with endogenous λ assuming a hump shaped Π. This example exhibits two interior equilibria, but only the one with the higher λ is stable. Notice that $\lambda = 0$ is also an equilibrium.

Proposition 3 states that, with the exception of Region 1, the sign of $\Pi'(\lambda)$ is not possible to characterize.[20] However, our hybrid model of direct and indirect asset liquidity introduces an endogenous force whereby Π can be decreasing in λ. This is important because it implies that our model can generate stable interior equilibria. Notice that this is not possible in LPW, where Π is a strictly increasing function.[21] We illustrate this new property of the model through a number of examples.

Figure 8 illustrates three different possibilities for $\Pi(\lambda)$, for a relatively low asset supply, i.e., $A < z(q^*)/2$. Case 1 represents the case where equilibrium lies in Region 2 for all $\lambda \in [0, 1]$. In this case, the negative effect of λ on Π is dominant and Π is strictly decreasing in its entire domain. Assuming that κ attains an intermediate value, as in the figure, this case implies a *unique and stable* interior equilibrium λ. Case 2 is similar in that equilibrium always lies within Region 2. However, this case assumes a higher asset supply, which increases the magnitude of the positive force of λ on Π (with a more plentiful asset supply, the seller who does not invest has more to lose). As a result, Π is now hump-shaped. With an intermediate κ, like the one indicated in the figure, we may now have two interior equilibria, but only the one that involves a higher λ is stable. Case 3 represents an example where increasing λ causes equilibrium to switch from Region 1 to Region 2. In this example, the positive effect of λ on Π is dominant, leading to an increasing profit function, which, in turn, implies an unstable interior equilibrium, as in LPW.

Figure 9 illustrates the case of plentiful supply, i.e., $A > z(q^*)/2$. Here, as λ increases equilibrium switches through Regions 1, 4, and 3. The slope of Π in Region 3 is negative making it hump-shaped. Notice that, within Region 3, Π becomes steeper as α increases. This is because the novel channel of our model (that tends to make $\Pi(\lambda)$ decreasing) works through the OTC market: sellers who choose to not invest in the information technology are better off with a high λ, if the buyers they meet were able to trade in the OTC market. But for this channel to be effective, matching in the OTC must be efficient, i.e., α must be large. As in Case 2 of Figure 8, here we can also have two interior equlibria, but only the one that involves the higher λ is stable.

The ability of our model to deliver stable interior equilibria is also important for comparative

[20] To be precise, this does not just mean that it is hard to tell whether it is positive or negative; it means that it could be first positive and then negative or vice versa even within the same region, let alone as equilibrium switches to different regions, as λ increases.

[21] LPW also suggest that interior equilibria are the most interesting. It is precisely because the interior equilibrium in that paper is unstable, that the authors explore an extension of the model, where the information cost is different for each seller. With a properly chosen distribution of costs among sellers, LPW can generate stable interior equilibria. We do not wish to claim that heterogeneous costs are unrealistic (perhaps it is quite the opposite). The point made here is that in our model interior equilibria are stable under a wider range of parameters than LPW, because of the novel channel introduced in our framework.

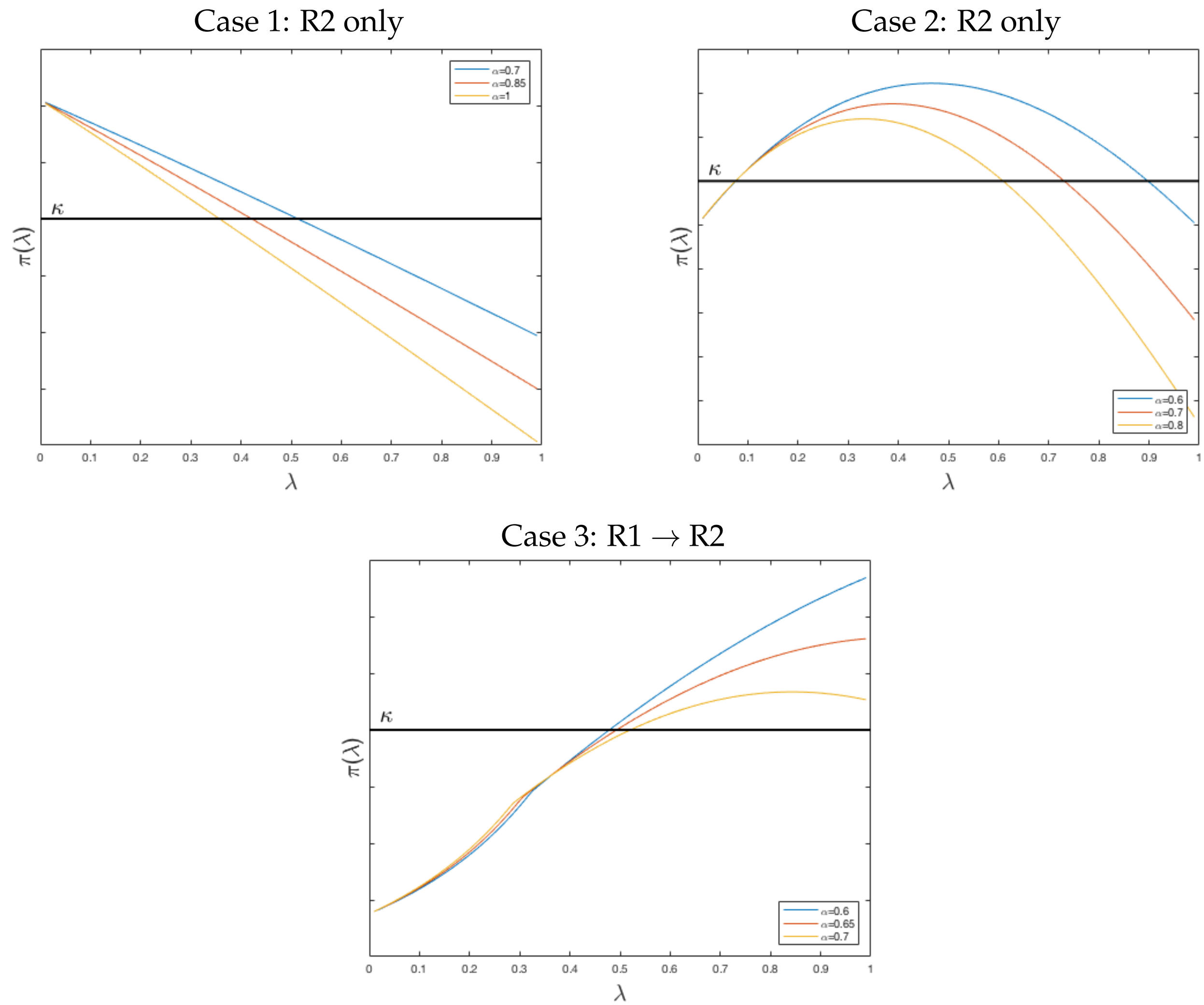


Figure 8: The profit function $\Pi(\lambda)$ under $A < z(q^*)/2$. All three cases assume $u(x) = x^{1-\rho}/(1-\rho)$, $\beta = 0.97$, $\rho = 0.5$, and $\eta = 0.5$. The parameters μ and A differ, thus, giving Π a different shape in each case. In particular, in Case 1, we have $\mu = 0.001, A = 0.004$; in Case 2, $\mu = 0.001, A = 0.1$; in Case 3, $\mu = 0.04, A = 0.4$.

statics exercises. Suppose we are interested in how changes in κ affect equilibrium λ. Around the stable interior equilibrium (e.g., Cases 1,2, or 4 in Figures 8 and 9), an increase in κ would lead to a decrease in the number of sellers who accept assets, as intuition suggests. This is not true for unstable equilibria: in Case 3 of Figure 8 (which is qualitatively equivalent to the model of LPW), an increase in κ would imply a higher λ, a logically inconsistent result. Thus, incorporating the notion of *indirect liquidity* into the model is not only empirically relevant, but it also generates a novel channel that improves the theoretical properties of equilibrium.

Inspection of Figures 8 and 9 reveals another noteworthy result: a more efficient secondary market, i.e., a higher α, reduces the measure of sellers who acquire information, around the stable interior equilibrium (when one exists). Intuitively, a more efficient OTC market allows

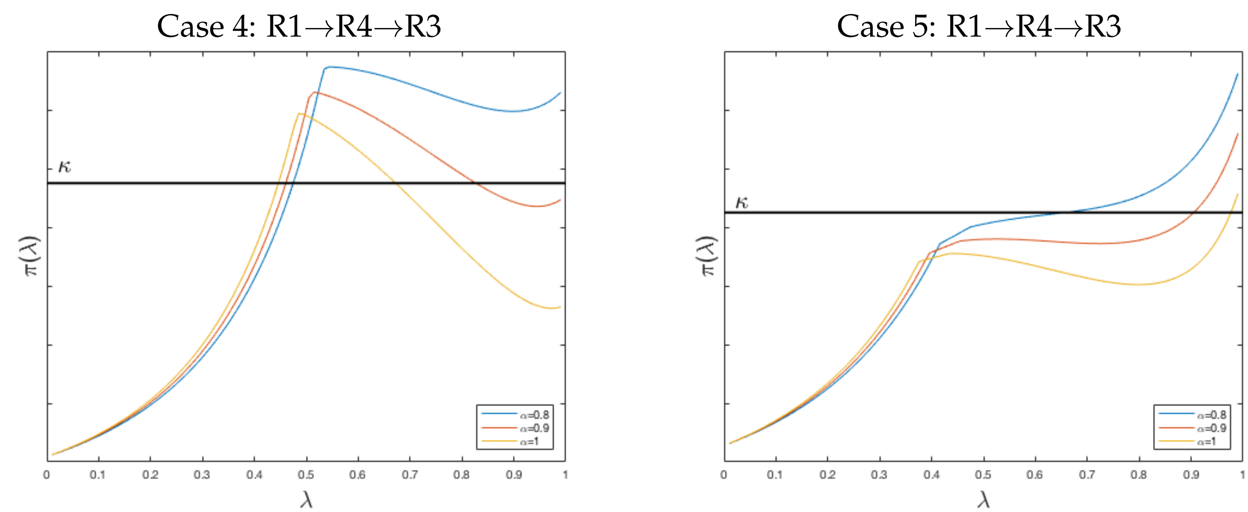


Figure 9: The profit function $\Pi(\lambda)$ under $A > z(q^*)/2$. Here, we have $u(x) = x^{1-\rho}/(1-\rho)$, $\beta = 0.97$, $\eta = 0.3$, and $\mu = 0.03$. The only difference between the two cases is that, in Case 4, we have $\rho = 0.62$ and $A = 1.08$, while in Case 5, we have $\rho = 0.5$ and $A = 0.9$.

buyers to take advantage of the indirect liquidity properties of assets, thus, reducing sellers' incentives to invest in the information that makes assets directly liquid. In a sense, the property of assets to be indirectly liquid through the OTC market serves as a substitute to direct liquidity. In Section 4.4, we will see that, while intuitive, this argument is not always accurate.

4.3 Aggregate Equilibrium with Endogenous λ

So far, this section has focused exclusively on the endogenous determination of λ. A steady state equilibrium for the generalized model is still described by Definition 1 and characterized by Propositions 1 and 2, except that, now, the asset acceptance rate λ is not exogenously given, but it is described by Lemma 7. Moreover, combining the comparative statics results of Section 4.2 with Lemma 6 allows us to study how changes in fundamental parameters, such as κ, affect important equilibrium variables, such as ψ. Specifically, recall from Lemma 6 that changes in λ effectively mimic changes in μ. Thus, an increase in κ, which decreases λ in the stable interior equilibrium, will result in a decrease in the asset price, ψ.

4.4 Welfare and Policy Implications

Policy makers and financial regulators are concerned about liquidity in certain markets, and often suggest legislations that will "improve liquidity". A natural question that arises is "What is the best way to improve asset liquidity?" Our model shows there are (at least) two ways to do this. First, the authorities could try to increase efficiency in secondary markets (in our model, an increase in α). For instance, they could bring forth legislations that promote more efficient/well-networked interdealer markets. Alternatively, they could subsidize agents for the cost of learning about the quality of the asset (in our model, a decrease in κ). A naive observer may suggest that these two ways of improving liquidity are virtually equivalent, and one should not worry too much about the details:[22] An increase in α (an improvement of indirect asset liquidity) or a decrease in κ (an improvement of direct asset liquidity) should have a similar effect on welfare, and, intuitively, that effect is expected to be positive.

The analysis of this section indicates that this reasoning is wrong.[23] To see why this is true, we first derive the steady state welfare function for this economy, given by

$$\begin{aligned}\mathcal{W} = \alpha\lambda(1-\lambda)\left[u(q_1^m) - q_1^m\right] + (1-\alpha\lambda)(1-\lambda)\left[u(q_1^n) - q_1^n\right] \\ + \lambda\left[u(q_2) - q_2\right] - \lambda\kappa. \qquad (23)\end{aligned}$$

The details of this derivation are relegated to Appendix A.3. Intuitively, the first line represents the DM surplus generated in type-1 meetings, depending on whether the buyer was matched or not in the OTC; the first term in the second line stands for the DM surplus in type-2 meetings; finally, the second term in the second line is the information cost paid by a measure λ of sellers.

We begin by examining the effect of an increase in α on welfare. One may conjecture that a higher efficiency in the OTC market should be welfare improving; after all, the OTC market helps allocate money into the hands of the agents who need it most (i.e. type 1 buyers). It turns out that this is questionable. Recall that a higher α also leads to a lower λ, which means more type-1 buyers in equilibrium. Thus, even though the matching process has become more efficient, there are more type 1-buyers trying to match with fewer type-2 buyers. Whether the matching probability of type-1 buyers in the OTC goes up or down depends on the elasticity of λ with respect to α. There is another, more subtle, force that tends to make $\mathcal{W}$ decreasing in α. When α is high, agents expect that it will be easy to acquire money in the OTC, thus, they carry less money in the CM.[24] This depresses the money demand and, consequently, the value of money and the surplus generated in all type-1 meetings.

Summing up, a change in α affects directly the OTC matching efficiency, but also indirectly the measure of the various types of meetings and the demand for money (i.e., it affects every

[22] One may say that some of the results of the paper so far point to that direction. For example, we saw that $\partial\lambda/\partial\alpha < 0$. That is, when the efficiency of matching in the OTC and, hence, the indirect asset liquidity improves, the role for direct asset liquidity (measured as the fraction of sellers who accept assets as MOE) decreases. In that sense, direct and indirect asset liquidity behave as substitutes.

[23] We remain agnostic as to how costly it is for the authorities to increase α or decrease κ; the goal of this section is not to deliver precise policy recommendations, but to highlight that the various policy options for "improving asset liquidity" can have very different effects on welfare.

[24] To be precise, an increase in α has two effects on money balances. The first is the one just described tending to decrease money balances. The second works through λ: a higher α will decrease λ and, as we know from Lemma 6, this will tend to increase money balances. Given the complexity of the model, it is impossible to check analytically whether an increase α will increase or decrease money balances.

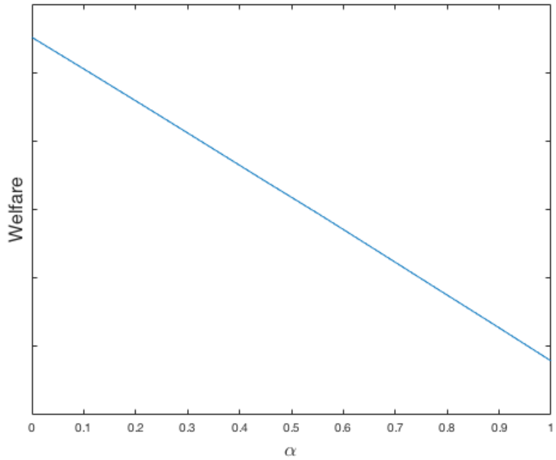


Figure 10: Welfare effects of α. Aggregate equilibrium is confined to R2 in this case. Here, we have $u(x) = x^{1-\rho}/(1-\rho)$, $\beta = 0.97$, $\rho = 0.5$, and $\eta = 0.5$, $\mu = 0.02$, $A = 0.0005$, and $\kappa = 0.000025$.

term in equation (23), except κ). The main result is that the sign of $d\mathcal{W}/d\alpha$ can go either way, depending on parameter values. Figure 10 represents the case where $d\mathcal{W}/d\alpha < 0$ for all α's: not only an increase in α could hurt welfare but, for certain parameter values, the economy would be better off if the secondary market *did not exist*, which is equivalent to $\alpha = 0$. But our model with $\alpha = 0$ *is* the model of LPW. This is yet another illustration—perhaps the most prominent one—of the striking new insights one obtains by adding a secondary market to that model. (Insights that go far beyond the fact that adding such a market is more 'realistic'.) The existence of a secondary market increases the measure of type-1 meetings (that is, it diminishes the asset's direct liquidity) *and at the same time* it decreases the equilibrium quantity of the only object that can serve as a MOE in these meetings, i.e., money, eventually hurting welfare.[25]

Next, consider the effect of changes in κ on welfare. Again, let us start with a conjecture. It seems reasonable that a lower κ would increase the fraction of sellers who recognize assets (hence, the asset's direct liquidity), and this should be welfare-improving, for two reasons. First, and more obviously, now assets facilitate transactions in more meetings. Second, with a higher λ type-1 buyers are fewer, and they should have an easier time matching in the OTC and proceeding to the DM with more cash. These arguments are valid but, once again, there is a drawback. The higher equilibrium λ (associated with a lower κ) decreases real balances (Lemma 6), generating a force that reduces welfare. In a number of simulations, we find that, typically, $d\mathcal{W}/d\kappa < 0$,

[25] As mentioned in Section 1.1, this is not the first paper on indirect liquidity, and some existing papers discuss the idea that welfare may decrease as agents gain easier access to the secondary market; see for example Berentsen et al. (2014) and Geromichalos and Herrenbrueck (2016). In those papers, the result occurs because agents believe they will have an easier time acquiring money in the secondary market, thus, decreasing their (*ex ante*) money demand. Here, this result manifests itself through more involved channels, in general equilibrium. A higher α affects the money demand, but also the measure of DM meetings where money is necessary as a MOE, which here is *endogenous*. This is because our model *is* the first model that combines direct and indirect liquidity.

i.e., a higher information cost hurts welfare, as expected and as depicted on the left panel of Figure 11. However, it is possible to find parameters for which $d\mathcal{W}/d\kappa > 0$, as in the right panel of the figure. Generally, $\mathcal{W}$ is likely to have an increasing segment if κ is high and μ is low. This is because, the money demand effect is especially strong when λ is low (that is when we have many type-1 meetings where money is the sole MOE), and this is likely to occur when κ (μ) is high (low). Within this parameter range, a further increase in κ could boost equilibrium real money balances by an amount large enough to generate a positive overall effect on welfare.

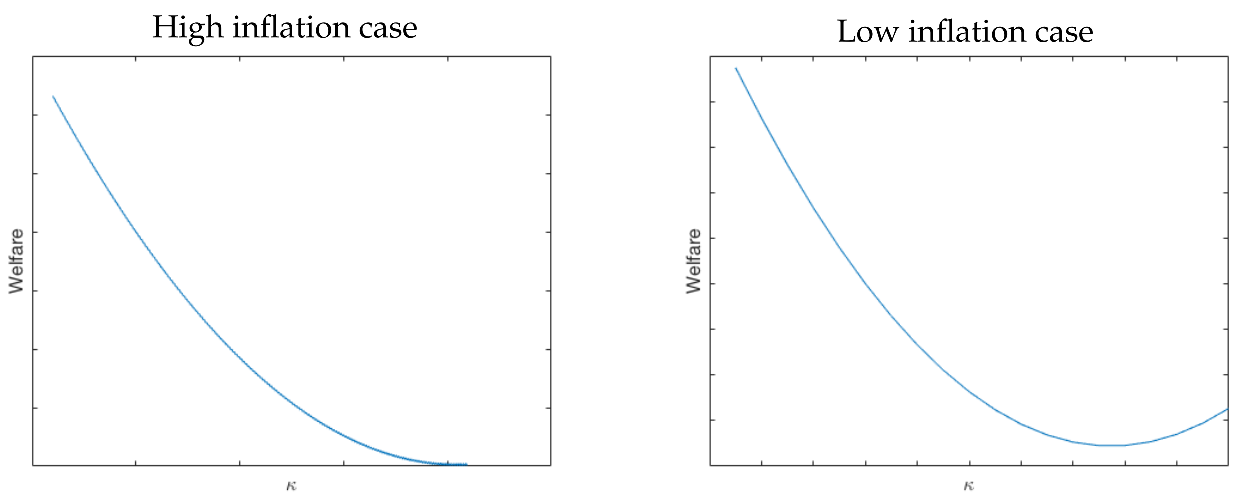


Figure 11: Welfare effects of κ. Aggregate equilibrium is confined to R2 in both cases. In both panels, we have $u(x) = ax^{1-\rho}/(1-\rho)$, $\beta = 0.97$, $\eta = 0.5$, $a = 2$, and $\rho = 0.6$, $\alpha = 1$, and $A = 0.01$. On the left panel, we have $\mu = 0.03$, on the right, $\mu = 0.02$.

Summing up, an increase in α (effectively an increase in indirect liquidity) and a decrease in κ (effectively an increase in direct liquidity) generate multiple, opposing forces in general equilibrium, and can have very different effects on welfare. Typically, an increase in direct liquidity, achieved by a reduction in κ and manifested by a high equilibrium λ will enhance welfare (although possible exceptions have been discussed). On the other hand, an increase in indirect liquidity, captured by an improvement in the OTC market efficiency, is likely to reduce equilibrium welfare. Sometimes, this effect can be so strong that the economy would be better off if the secondary asset market ceased to exist altogether. Of course, that is not to say that secondary asset markets are always bad for welfare. For one, trade in secondary asset markets does not only take place for liquidity motives, as our model assumes.[26] What the model does say is that, from the policy perspective, it is better to enhance the direct rather than indirect liquidity of assets, i.e., to promote an environment where assets serve directly as means of payment. This could be achieved by improving financial literacy among economic agents and reducing the information

[26] For example, Wang (2019) extends Geromichalos and Herrenbrueck (2016) by assuming that some agents trade due to liquidity needs and some others driven by private information about the returns of certain assets.

asymmetry about asset returns.

5 Conclusion

By now, a large body of literature documents that many assets are priced not only for their fundamental value (roughly defined as the present value of all future payments), but also for the liquidity services these assets may provide before their maturity date.[27] Agents are willing to buy assets at a *liquidity premium* mainly for two reasons: i) they expect to use them directly as facilitators of trade, i.e., media of exchange or collateral, in transactions; or ii) they expect to be able to sell them for money in a secondary market upon the arrival of a liquidity need. These two types of liquidity, which in our paper have been dubbed "direct" and "indirect", respectively, have been studied extensively in the literature, but only in *isolation*. Our paper demonstrates that this practice is not without loss of generality. Whether a buyer needs to visit a secondary market in order to 'liquidate' assets for cash, depends on whether the seller of the goods/services she wishes to purchase will accept these assets as payment. Vice versa, the willingness of a seller of goods/services to acquire information that allows her to recognize and accept assets (other than money) in transactions, depends on the existence and efficiency of a secondary market where the buyer could sell her assets for cash.[28]

Our model encompasses both of these notions of liquidity and determines their relative importance endogenously, as a function of two fundamental parameters: i) The information cost that sellers must incur in order to recognize and accept assets in transactions; and ii) The efficiency of matching in the secondary asset market. Given these parameters, we study asset prices and how these prices are affected by monetary policy. Our model formalizes some intuitive ideas, but also delivers some new and surprising insights. As an example of the former, we show that a higher secondary market efficiency increases an asset's indirect liquidity and crowds out its direct liquidity, i.e., it discourages sellers from acquiring information and accepting assets as payment. As an example of the latter, we show that changes in direct and indirect liquidity have very different effects on welfare. An authority whose objective is to maximize welfare has better chances of achieving this goal by promoting direct liquidity, i.e., by incentivizing sellers to get informed about the value/quality of assets and accept them directly as means of payment. And, a more efficient secondary market can crowd out the asset's direct liquidity to such an extent, that the economy may be better off if that market ceased to exist.

[27] See for example Krishnamurthy and Vissing-Jorgensen (2012) for the case of Treasury bonds.

[28] If I am a buyer, why would I waste time and resources to visit a secondary market and sell assets for money, when I know that the seller of the commodity I wish to buy will accept these assets as a medium of exchange? If I am a seller, why would I pay for information that allows me to recognize complex financial instruments, when I know that buyers have access to an ultra-liquid secondary market, where these securities can be turned into money quickly and at minimal cost?

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A Appendix

A.1 The Demand for Money

The demand for money is plotted in the lower panel of Figure 12 as a function of the holding cost of money, $\varphi/(\beta\hat{\varphi})$, and for two levels of asset holdings, $\hat{a} = \{a^l, a^h\}$, with $a^h > a^l$ (indicated in the upper panel).[29] Generally, the money demand of a buyer with low asset holdings (D^l_m)

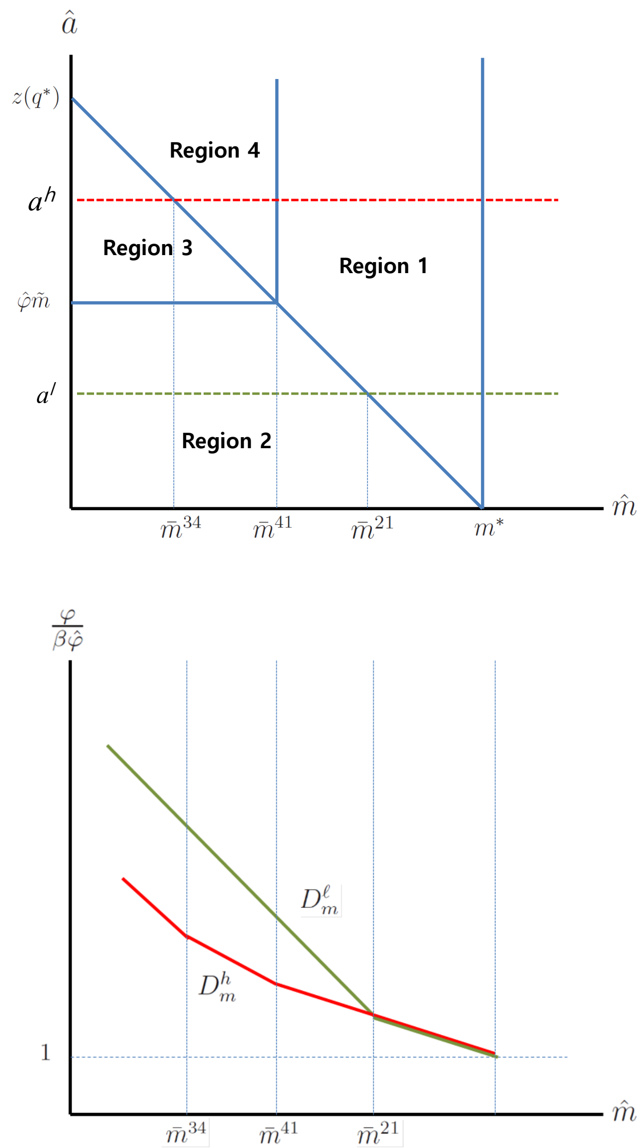

Figure 12: Money demand given high and low asset holdings.

[29] More precisely, the holding cost of money is $\varphi/(\beta\hat{\varphi}) - 1$. In steady state, and exploiting the Fisher equation, one can easily show that this expression is equal to i, the nominal interest rate on a perfectly illiquid bond. For a more detailed discussion on the derivation of this interest rate and the importance of measuring it properly in the data, see Geromichalos and Herrenbrueck (2017).

will lie above that of a buyer with high asset holdings (D_m^h), because the former must rely more heavily on her money for liquidity. Moreover, the only region in which D_m^l and D_m^h coincide is Region 1. This is because in Region 1 an additional dollar is good only for an unmatched type 1 buyer, and for that buyer asset holdings are irrelevant: she cannot use them for direct liquidity because she is a type 1, and she cannot use them for indirect liquidity because she is unmatched (in the OTC). Finally, as $\hat{m}$ decreases, the buyer's valuation for an additional dollar increases. This can be seen from the negative slope of the demand curves, and the fact that as we move from Region 1 into regions where liquid assets as scarcer (Regions 4 and 3, for $\hat{a} = a^h$, and Region 2, for $\hat{a} = a^l$), the demand curves become steeper.[30]

A.2 Derivation of Figure 4

Figure 13 describes the aggregate regions, unlike Figure 2 which shows the regions of a representative buyer. Notice that the symmetry of the equilibrium excludes the case where type 1 and type 2 buyers hold different portfolios, because they are identical *ex* ante. As a result, the flat line between Region 3 and 2 which the representative buyer faces becomes a 45 degree line in the symmetric equilibrium. Also, the y-axis intercept which the flat line touches on is now equal to $\frac{1}{2}z(q^*)$.

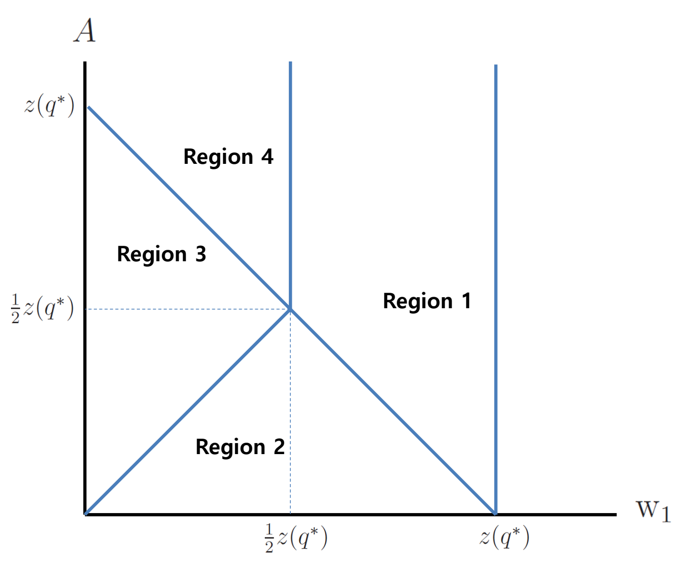

Figure 13: Aggregate regions of equilibrium, in terms of money and asset supply.

In order to examine how the equilibrium objects such as the asset price and the trade volume

[30] As a more detailed example, consider D_m^h in a neighborhood of $\bar{m}^{34}$ (the level of $\hat{m}$ that brings the agent on the boundary of Regions 3 and 4, for $\hat{a} = a^h$). As the figure reveals, D_m^h is steeper for $\hat{m} < \bar{m}^{34}$, because an additional unit of money in Region 3 increases the buyer's DM consumption irrespectively of her type (in this region, we have $\hat{m} + \tilde{m} < m^*$and $\hat{\varphi}\hat{m} + \hat{a} < z(q^*)$). On the contrary, in Region 4 (i.e., for $\hat{m} > \bar{m}^{34}$), an additional dollar will only increase the consumption of a type 1 buyer (in this region, we have $\hat{m} + \tilde{m} < m^*$, but $\hat{\varphi}\hat{m} + \hat{a} \geq z(q^*)$).

of DM goods respond to changes in the asset supply, A, and the money growth rate as a policy parameter, μ, we need to divide the equilibrium aggregate regions on the asset supply and the money growth rate coordinates, instead of the asset supply and the real balances. The following lemma explains how the equilibrium regions are divided by the asset supply, A, and the money growth rate, μ.

Lemma 8. *There exist cutoffs $\bar{\mu} > \beta - 1$ as follows.*
Case 1: $z(q^*) \leq A$,

- *i. If $\mu \in (\beta - 1, \bar{\mu}_{14})$, equilibrium is in the interior of Region 1,*
- *ii. If $\mu \in (\bar{\mu}_{14}, \infty)$, equilibrium is in the interior of Region 4.*

Case 2: $\frac{z(q^*)}{2} < A < z(q^*)$,

- *i. If $\mu \in (\beta - 1, \bar{\mu}_{14})$, equilibrium is in the interior of Region 1,*
- *ii. If $\mu \in (\bar{\mu}_{14}, \bar{\mu}_{43})$, equilibrium is in the interior of Region 4,*
- *iii. If $\mu \in (\bar{\mu}_{43}, \infty)$, equilibrium is in the interior of Region 3.*

Case 3: $A < \frac{z(q^*)}{2}$,

- *i. If $\mu \in (\beta - 1, \bar{\mu}_{12})$, equilibrium is in the interior of Region 1,*
- *ii. If $\mu \in (\bar{\mu}_{12}, \bar{\mu}_{23})$, equilibrium is in the interior of Region 2,*
- *iii. If $\mu \in (\bar{\mu}_{23}, \infty)$, equilibrium is in the interior of Region 3.*

Proof. Consider three cases, which are divided by the level of aggregate asset supply.

i. $z(q^*) \leq A$. Agents are in Regions 1 and 4 because $\hat{a} = A$ in the equilibrium. The money demand in these regions (equations (16) and (22)) are continuous within regions as well as across the boundary between them. Also, the demand is strictly decreasing in μ. Hence, there exists a $\bar{\mu}_{14}$.

ii. $\frac{z(q^*)}{2} < A < z(q^*)$. Agents can be in Region 1, 4, and 3. It depends on their real balances. If the real balances are high enough, they will be in Region 1. As the real balances decrease, they will move to Region 4 and then Region 3. The money demand in these regions (equations (16), (22) and (20)) are continuous within regions as well as across the boundary between them. In addition, it is also strictly decreasing in μ. Hence, there exist a $\bar{\mu}_{14}$ a $\bar{\mu}_{43}$.

iii. $A < \frac{z(q^*)}{2}$. Agents can be in Region 1, 2, and 3. It depends on their real balances. If the real balances are high enough, they will be in Region 1. As the real balances decrease, they will move to Region 2 and then Region 3. The money demand in these regions (equations (16), (18) and (20)) are continuous within regions as well as across the boundary between them. In addition, it is also strictly decreasing in μ. Hence, there exist a $\bar{\mu}_{12}$ a $\bar{\mu}_{23}$.

□

Figure 3 graphically shows Lemma 8: how the equilibrium regions are divided by the asset supply, A and the money growth rate, μ. The money demand stays low when the money growth rate is high. Since the higher money growth rate implies the higher opportunity cost of holding money, buyers become less willing to bring money to the DM for transactions. For this reason, regardless of the level of the asset supply, buyers relocates to regions where their money holdings in the DM are relatively scarce, when the money growth rate gets high. One of the differences between high and low asset supply is whether the buyers pass through Region 4 or Region 2 when the money growth rate is, roughly speaking, at moderate levels. If the asset supply is scarce, they take up Region 2. Lastly, if the asset supply is exactly equal to $z(q^*)/2$), then buyers find themselves in either Region 1 or Region 3, depending on the value of μ.

A.3 Derivation of the $\mathcal{W}$ function in Section 4.4

Note that the equilibrium CM consumption and work effort will differ among agents with different trading histories. For instance, a type-1 buyer who traded in the OTC carries less money than a type-1 buyer who did not match with anyone, so the former will have to work harder to rebalance her portfolio. Moreover, the equilibrium CM consumption and work hours will also differ among sellers, depending on their type and whether their trading partners in the DM have previously matched in the OTC or not. For instance, a type-1 seller who traded with a non-OTC matched type-1 buyer will enter the CM with fewer real balances than a seller who traded with a type-1 buyer who has successfully matched in the OTC. Therefore, there are potentially 8 different possibilities.

We divide the various possibilities as follows. First, we let X^k_{Bj} (H^k_{Bj}), $j \in \{1,2\}$ and $k \in \{m,n\}$ denote the equilibrium CM consumption (work effort) for the buyer type j whose previous OTC trading status was k (where m stands for a successful OTC match, while n does for an unsuccessful OTC match). Likewise, we let X^k_{Sj} (H^k_{Sj}), $j \in \{1,2\}$ and $k \in \{m,n\}$ denote the equilibrium CM consumption (work effort) of the seller type j who matched with a type j buyer, whose OTC trading status is k.

Let $\mathcal{C}_C$ denote the total net CM utilities of (all) agents. Then, we obtain

$$\begin{aligned}\mathcal{C}_C =& \alpha\lambda(1-\lambda)(X^m_{B1} - H^m_{B1}) + (1-\alpha\lambda)(1-\lambda)(X^n_{B1} - H^n_{B1}) \\ &+ \alpha(1-\lambda)\lambda(X^m_{B2} - H^m_{B2}) + [1-\alpha(1-\lambda)]\lambda(X^n_{B2} - H^n_{B2}) \\ &+ \alpha\lambda(1-\lambda)X^m_{S1} + (1-\alpha\lambda)(1-\lambda)X^n_{S1} + \alpha(1-\lambda)\lambda X^m_{S2} + [1-\alpha(1-\lambda)]\lambda X^n_{S2}.\end{aligned}$$

Note that sellers' work effort (H^k_{Sj}) always equal zero at the steady state equilibrium. This explains the third line in above equation. The followings are the breakdown of X^k_{Bj} and H^k_{Bj} for different agents with different trading histories.

Matched type-1 buyers and sellers

If $\chi = A$ or $A < \bar{a}(\mathrm{w}_1)$, i.e., Region 2 only, then,

$$X^m_{B1} = \max\{0, \mathrm{w}_1 + A - z(q^*)\} \;\; H^m_{B1} = \mathrm{w}_1 + A, \;\text{ and }\; X^m_{S1} = \min\{\mathrm{w}_1 + A, z(q^*)\},$$

If $\chi < A$ or $A > \bar{a}(\mathrm{w}_1)$, i.e., Region 1, 3, and 4, then,

$$X^m_{B1} = A - \bar{a}(\mathrm{w}_1) \;\; H^m_{B1} = \min\{\mathrm{w}_1 + A, z(q^*)\}, \;\text{ and }\; X^m_{S1} = \min\{\mathrm{w}_1 + \bar{a}(\mathrm{w}_1), z(q^*)\},$$

Therefore, the net CM utilities for these particular agents sum up to

$$\alpha\lambda(1-\lambda)(X^m_{B1} - H^m_{B1}) + \alpha\lambda(1-\lambda)X^m_{S1} = 0. \tag{a.1}$$

Matched type-2 buyers and sellers

$$\begin{aligned}
X^m_{B2} &= \begin{cases} \mathrm{w}_1 + A - z(q^*), & \text{if } z(q^*) \leq \mathrm{w}_1 + A, \\ 0, & \text{if } z(q^*) > \mathrm{w}_1 + A, \end{cases} \\
H^m_{B2} &= \begin{cases} \mathrm{w}_1 + A, & \text{if } z(q^*) \leq \mathrm{w}_1 + A, \\ \mathrm{w}_1 + A, & \text{if } z(q^*) > \mathrm{w}_1 + A, \end{cases} \\
H^m_{B2} &= \begin{cases} z(q^*), & \text{if } z(q^*) \leq \mathrm{w}_1 + A, \\ \mathrm{w}_1 + A, & \text{if } z(q^*) > \mathrm{w}_1 + A. \end{cases}
\end{aligned}$$

The net CM utilities for OTC-matched-type-2 buyer and type-2 sellers sum up to

$$\alpha(1-\lambda)\lambda\left\{X^m_{B2} - H^m_{B2} + X^m_{S2}\right\} = 0. \tag{a.2}$$

Unmatched type-1 buyers and sellers
Under this case,

$$X^n_{B1} = A \;\; H^n_{B1} = \mathrm{w}_1 + A, \;\text{ and }\; X^m_{S1} = \mathrm{w}_1,$$

Therefore, the net CM utilities for these agents sum up to

$$(1-\alpha\lambda)(1-\lambda)\left\{X^n_{B1} - H^n_{B1} + X^n_{S1}\right\} = 0. \tag{a.3}$$

Unmatched type-2 buyers and sellers
The net CM utilities for OTC-unmatched-type-2 buyers and sellers sum up to

$$[1-\alpha(1-\lambda)]\lambda\left\{X^m_{B2} - H^m_{B2} + X^m_{S2}\right\} = 0. \tag{a.4}$$

Combining (a.1), (a.2), (a.3), and (a.4) leads to $\mathcal{C}_C = 0$. If one adds total net DM utilities and the information acquisition cost term $\lambda\kappa$, one finally gets (23).

A.4 Proof of Statements

Proof. Proof of Lemma 1.

If we take a derivative of the objective function $\eta\{u(q_i) - q_i\}$ in (7) with respect to q_i, the first order condition is that $u'(q_i) = 1$. It implies that the objective function is maximized at $q_i = q^*$ for $i \in \{1, 2\}$. First, consider a Type 1 meeting, where the real balances of a buyer for trade are only determined by his/her money holdings, m. As long as the real money balances, φm, are equal to or greater than $z(q^*)$, q_1 will be equal to the first best quantity, q^*, and the buyer will hand over only m^*, i.e., $d_1^m = m^*$ by the proportional bargaining constraint. It is obvious that $d_1^a = 0$ because a Type 1 seller does not accept real assets as a MOE. On the other hand, the real money balances are strictly less than $z(q^*)$, the buyer gives up all his/her money in order to increase the total surplus, $u(q_1) - q_1$, as much as possible, i.e., $d_1^m = m$, and q_1 is equal to the corresponding amount to the real money balances, $z^{-1}(\varphi m)$. Second, in a Type 2 meeting, the real balances for trade are determined by not only m but also a. Similarly, if the real balances, $\varphi m + a$, are equal to or greater than $z(q^*)$, $q_2 = q^*$, and otherwise $q_2 = z^{-1}(\varphi m + a)$. In the former case, the total real balances that a buyer hand over are exactly equal to $z(q^*)$, but d_2^m and d_2^a are indeterminate only if $\varphi d_2^m + d_2^a = z(q^*)$ by the proportional bargaining constraint because they are perfect substitutes to each other and only the total real value transferred matters. In the latter case, $d_2^m = m$ and $d_2^a = a$ for the same reason in the Type 1 meeting. □

Proof. Proof of Lemma 2.

The OTC bargaining problem simplifies to

$$\max_{\chi,\zeta} S_1 \ s.t. \ S_2 = 0, \ \chi \leq a, \ \zeta \leq \tilde{m}, \text{ and } \zeta = p\chi,$$

where
$$\begin{aligned} S_1 =& V_1(m+\zeta, a-\chi) - V_1(m,a) = u(q_1(m+\zeta)) - u(q_1(m)) + \varphi[d_1(m) - d_1(m+\zeta)] + \varphi\zeta - \chi \\ =& u(q_1(m+\zeta)) - u(q_1(m)) - \chi. \end{aligned}$$

The last equality comes from the fact that any trade that would make $m + \zeta > m^*$ would not generate surplus, thereby $d(m) = m$. Finally, S_1 can be expressed as follows.

$$S_1 = u(\tilde{q}_1(m+\zeta)) - u(\tilde{q}_1(m)) - \chi.$$

where $\tilde{q}(m) \equiv \{q : \varphi m = z(q)\}$. Similarly, S_2 can be expressed by

$$\begin{aligned} S_2 =& V_2(\tilde{m}-\zeta, \tilde{a}+\chi) - V_2(\tilde{m}, \tilde{a}) \\ =& u(q_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - u(q_2(\tilde{m}, \tilde{a})) - \varphi\zeta \\ & - \varphi d_2^m(\tilde{m}-\zeta, \tilde{a}+\chi) + \chi - d_2^a(\tilde{m}-\zeta, \tilde{a}+\chi) + \varphi d_2^m(\tilde{m}, \tilde{a}) + d_2^a(\tilde{m}, \tilde{a}). \end{aligned}$$

Now, we will consider two different cases:

Case 1: $\varphi\tilde{m} + \tilde{a} \geq z(q^*)$

In this case, we are in Region 1. Our claim is that post-OTC trade balances must be greater than or equal to $z(q^*)$. We prove this claim by contradiction. Suppose not, i.e., $\varphi\tilde{m}+\tilde{a}+\chi-\varphi\zeta < z(q^*)$. Then, we are in binding DM branch such that

$$\begin{aligned} S_2 =& u(\tilde{q}_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - \varphi\zeta - \varphi\tilde{m} + \varphi\zeta + \chi - \tilde{a} - \chi - u(q^*) + z(q^*) \\ =& [u(\tilde{q}_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - u(q^*)] + [z(q^*) - (\varphi\tilde{m} + \tilde{a})] < 0. \end{aligned}$$

where the last inequality holds true since $\tilde{q}_2 < q^*$ in this region. Thus, it must be that $\varphi\tilde{m} + \tilde{a} + \chi - \varphi\zeta \geq z(q^*)$. Given that we must have $\tilde{q}_2(m+\zeta, a-\chi) = z(q^*)$, $S_2 = \chi - \varphi\zeta$. Thus, any solution must have $\chi = \varphi\zeta$, which also implies $p = 1/\varphi$. Now, the OTC bargaining problem is further simplified to

$$\max_{\zeta} u(\tilde{q}_1(m+\zeta)) - u(\tilde{q}_1(m)) - \varphi\zeta \ \ s.t.\ \zeta \leq \tilde{m},\ \ \chi \leq a.$$

FOC must be then $u'(\tilde{q})(d\tilde{q}_1/d\zeta) = \varphi$. But since $\varphi m + \varphi\zeta = z(\tilde{q}_1)$, $d\tilde{q}_1/d\zeta = \varphi/z'(\tilde{q}_1)$. Plugging the latter into the FOC should yield $\tilde{q}_1 = q^*$.

Case 2: $\varphi\tilde{m} + \tilde{a} < z(q^*)$

Similar to the case 1, we also claim that $\varphi\tilde{m} + \tilde{a} + \chi - \varphi\zeta < z(q^*)$. Suppose not. Then,

$$\begin{aligned} S_2 =& u(q^*) - z(q^*) - u(\tilde{q}_2(\tilde{m}, \tilde{a})) - \varphi\zeta + \chi + \varphi\tilde{m} + \tilde{a} \\ =& [u(q^*) - u(\tilde{q}_2(\tilde{m}, \tilde{a}))] + [\varphi\tilde{m} - \varphi\zeta + \tilde{a} + \chi - z(q^*)] > 0, \end{aligned}$$

which is a contradiction. So, $\tilde{q}_2 < q^*$ in this case and $S_2 = u(\tilde{q}_2(\tilde{m}-\zeta, \tilde{a}+\chi)) - u(\tilde{q}_2(\tilde{m}, \tilde{a}))$. Since $S_2 = 0$, it must be that $\chi = \varphi\zeta$.

Given solutions in the case 1 and 2, the rest of the proof goes as follows. First, in the case 1 type 1 buyer always wants to set $\tilde{q}_1 = q^*$. Yet, there are 2 reasons why that might not be possible. First, if a is unlimited and $\tilde{m}$ is limited in the sense that $m + \tilde{m} < m^*$, then necessarily $\zeta = \tilde{m}$ and $\chi = \varphi\tilde{m}$. Here, the unlimited a means $a \geq \varphi\tilde{m}$. If $m + \tilde{m} \geq m^*$ and a is limited then, $a = \chi$ and the type 1 cannot get the 1st best, i.e., $a = \varphi\zeta < z(q^*) - \varphi m$. In the case 2, if $m + \tilde{m} \geq m^*$ but $a < z(q^*) - \varphi m$, then $\chi = a$ and $\zeta = a/\varphi$. On the other hand, if $m + \tilde{m} < m^*$ and $a \geq \varphi\tilde{m}$, then $\zeta = \tilde{m}$ and $\chi = \varphi\tilde{m}$. □

Proof. Proof of Lemma 3.

First, we describe how to derive (16) to (22) from (15). We substitute the bargaining solutions in each region in Lemma 2 into (15), and then we take a derivative of J in each region with respect to $\hat{m}$ and $\hat{a}$, respectively, in order to obtain (16) to (22). For example, $\chi = \hat{\varphi}(m^* - \hat{m})$,

$p\chi = m^* - \hat{m}$, and $\rho = z(q^*)$ in Region 1. Plugging these solutions into J yields

$$\begin{aligned} J^1(\hat{m}, \hat{a}) = & -\varphi\hat{m} - \psi\hat{a} \\ & + \beta\big\{(1-\lambda)\big[\alpha\lambda\big(u(q^*)) + \hat{a} - \hat{\varphi}(m^* - \hat{m})\big) + (1-\alpha\lambda)\big(u(q_1(\hat{m})) + \hat{a}\big)\big] \\ & + \lambda\big[\big(u(q^*)) + \hat{\varphi}\hat{m} + \hat{a} - z(q^*)\big)\big]\big\}. \end{aligned}$$

Then it is straightforward to obtain J_1^1 and J_2^1 as show in (16) and (17). We also derive the other derivatives of J in the same way.

Now, we prove the three properties of J mentioned in the lemma.

i. The bargaining solutions and the constraints in the OTC are continuous. Hence, J is continuous. Also, it is differentiable all over the regions across boundaries between regions because J_j for $j \in \{1, 2\}$, is continuous across the boundaries: $J_{j+} = J_{j-}$.

ii. J_1 is continuous and strictly decreasing in $\hat{m}$ all over the regions, in particular, because $L(\varphi m)$ is strictly decreasing: $J_1^i < 0$ for $i \in \{1, 2, 3, 4\}$. Hence, it is strictly concave in $\hat{m}$. On the other hand, since J_2 is a constant in Region 1 and 4, and decreasing in $\hat{a}$ in Region 2 and 3: $J_2^i \leq 0$ for $i \in \{1, 2, 3, 4\}$. It is weakly concave in $\hat{a}$.

iii. It is easily proved from i and ii.

□

Proof. Proof of Lemma 4.

Given prices $(\varphi, \hat{\varphi}, \psi)$and beliefs $(\tilde{m}, \tilde{a})$,

i. since J is weakly concave and differentiable everywhere, the condition that $J_j^i(\hat{m}, \hat{a}) = 0$ must hold at the optimum;

ii. when $\psi = \beta$, the fact that $L(\cdot) > 0$ in Region 2 and 3 implies that $J_j^i(\hat{m}, \hat{a}) > 0$. Hence Region 2 and 3 are ruled out. In Region 1 and 4, J_1^j is strictly concave in $\hat{m}$, and so the optimal choice of $\hat{m}$ is uniquely determined by the condition that $J_1^i(\hat{m}, \hat{a}) = 0$ with satisfying the condition that $\varphi > \beta\hat{\varphi}$;

iii. When $\psi > \beta$, Region 1 and 4 are ruled out because the condition that $\psi = \beta$ must hold at the optimum. Moreover, since J_1^j is strictly concave in both $\hat{m}$ and $\hat{a}$ in Region 2 and 3, the optimal portfolio choice, $(\hat{m}, \hat{a})$, is uniquely determined the condition that $J_j^i(\hat{m}, \hat{a}) = 0$ with satisfying the condition that $\varphi > \beta\hat{\varphi}$.

□

Proof. Proof of Lemma 5.

Since the equilibrium objects $\{q_1^n, q_1^m, q_2^n, q_2^m, \chi, p\}$ are uniquely determined by $\mathrm{w}_1 = \hat{\varphi}(1 + \mu)M$, A and ψ, we need to show first that w_1 uniquely exists. Since we only take into account that $\mu > \beta - 1$, and $\varphi > \beta\hat{\varphi}$, an optimal choice of money $\hat{m}$ is uniquely determined by the first order conditions in Lemma 3. Then, φ $(= (1+\mu)\hat{\varphi})$ and $\hat{\varphi}$ should be set such that markets clear, i.e., $\hat{m} = (1+\mu)M$ and $\hat{a} = A$. As a result, w_1 uniquely exists. Moreover, ψ is uniquely pinned down such that the first order conditions in terms of asset holdings in Lemma 3 and $\hat{a} = A$. Hence, $\{q_1^n, q_1^m, q_2^n, q_2^m, \chi, p\}$ also exist and are unique, respectively. $\square$

Proof. Proof of Proposition 1.

Case 1: If $z(q^*) \leq A$, the equilibrium is located either in Region 1 or 4, depending on the level of the money growth rate, μ. (17) shows that the asset price equals the fundamental value: $\psi = \beta$.

Case 2: If $z(q^*)/2 < A < z(q^*)$, the equilibrium is in Region 1, 4, or 3.

i. Let $\beta - 1 < \mu \leq \bar{\mu}_{43}$. Then, the equilibrium is in Region 1, 4 or the boundary between them, and the asset price, ψ, is equal to β as in Case 1.

ii. Let $\bar{\mu}_{43} < \mu < \infty$. Then, the equilibrium is in Region 3, where $\psi = \beta\{(1-\lambda) + \lambda L(\varphi M + A)\}$. Since $L(\varphi M + A) > 1$ in Region 3, ψ is greater than the fundamental value, β. Moreover, higher μ decreases φM by (20), which leads to higher ψ because $L'(\cdot) < 0$. As a result, ψ strictly increases in μ: $\psi'(\mu) > 0$.

Case 3: If $A < z(q^*)/2$, the equilibrium is in Region 1, 2, or 3.

i. Let $\beta - 1 < \mu \leq \bar{\mu}_{12}$. Then, the equilibrium is in Region 1 or the boundary between Region 1 and 2, where the asset price, ψ, is equal to β by (17).

ii. Let $\bar{\mu}_{12} < \mu \leq \bar{\mu}_{23}$. Then, the equilibrium is in Region 2 or on the boundary between Regions 2 and 3, where $\psi = \beta\{(1-\lambda)[\alpha\lambda L(\varphi M + A) + (1-\alpha\lambda)] + \lambda L(\varphi M + A)\}$. Since $L(\varphi M + A) > 1$ in Region 2 and its boundary with Region 3, ψ is greater than β. In addition, higher μ decreases φM by (18), which results in higher ψ because $L'(\cdot) < 0$. Hence, ψ strictly increases in μ: $\psi'(\mu) > 0$. Lastly, let $\bar{\mu}_{23} < \mu \leq \infty$. Then, the equilibrium is in Region 3, where $\psi(\mu) > \beta$ and $\psi'(\mu) > 0$ as shown in ii. in Case 2.

$\square$

Proof. Proof of Proposition 2.

In equilibrium, $q_1^n = z^{-1}(\mathrm{w}_1)$, and w_1 strictly decreases in μ in all of the regions, according to the first order conditions with respect to $\hat{m}$ in Lemma 3. Since $z^{-1}(\cdot)$ is a strictly increasing function, q_1^n strictly decreases in μ. Moreover, q_1^m and q_2 weakly decrease in μ. According to Lemma 1, q_1^m and q_2 is equal to q^* or a strictly increasing function of w_1, which decreases in μ, depending on which region the equilibrium is located in. Consequently, since the average trade volume in the DM, Q, is a linear function of q_1^n, q_1^m and q_2, Q is a strictly decreasing function of μ for any $A \leq z(q^*)$. $\square$

Proof. Proof of Lemma 6

<u>Region 1:</u> From (16) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda + (1-\alpha\lambda)L(\mathrm{w}_1)] + \lambda.$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned} \frac{d\mathrm{w}_1}{d\lambda} &= \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)} \frac{L(\mathrm{w}_1)-1}{L'(\mathrm{w}_1)} < 0, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)\lambda(1-L(\mathrm{w}_1))}{(1-\lambda)(1-\alpha\lambda)L'(\mathrm{w}_1)} < 0. \end{aligned} \tag{a.5}$$

Region 2: From (18) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda L(\mathrm{w}_1+a) + (1-\alpha\lambda)L(\mathrm{w}_1)] + \lambda L(\mathrm{w}_1+a).$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned} \frac{d\mathrm{w}_1}{d\lambda} &= \frac{[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]+\alpha(1-\lambda)[L(\mathrm{w}_1+a)-L(\mathrm{w})]-\alpha\lambda[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]}{-(1-\lambda)[\alpha\lambda L'(\mathrm{w}_1+a)+(1-\alpha\lambda)L'(\mathrm{w})]-\lambda L'(\mathrm{w}_1+a)}, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)[\lambda L(\mathrm{w}_1+a)-\lambda L(\mathrm{w}_1)]}{(1-\lambda)[\alpha\lambda L'(\mathrm{w}_1+a)+(1-\alpha\lambda)L'(\mathrm{w}_1)]+\lambda L'(\mathrm{w}_1+a)} < 0. \end{aligned} \tag{a.6}$$

where (a.6) holds true since $L(\mathrm{w}_1+a)-L(\mathrm{w}_1) < \alpha\lambda[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]$.

Region 3: From (20) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda L(2\mathrm{w}_1) + (1-\alpha\lambda)L(\mathrm{w}_1)] + \lambda L(\mathrm{w}_1+a).$$

By the implicit function theorem, the following two must hold true.

$$\begin{aligned} \frac{d\mathrm{w}_1}{d\lambda} &= \frac{[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)]+\alpha(1-\lambda)[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]-\alpha\lambda[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]}{-(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1)+(1-\alpha\lambda)L'(\mathrm{w}_1)]-\lambda L'(\mathrm{w}_1+a)}, \\ \frac{d\mathrm{w}_1}{d\alpha} &= -\frac{(1-\lambda)[\lambda L(2\mathrm{w}_1)-\lambda L(\mathrm{w}_1)]}{(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1)+(1-\alpha\lambda)L'(\mathrm{w}_1)]+\lambda L'(\mathrm{w}_1+a)} < 0. \end{aligned} \tag{a.7}$$

where (a.7) holds true due to the following. Note that $a > \mathrm{w}_1$ in Region 3. Thus, $L(\mathrm{w}_1+a) < L(2\mathrm{w}_1)$. This leads to $L(\mathrm{w}_1+a)-L(\mathrm{w}_1) < L(2\mathrm{w}_1)-L(\mathrm{w}_1)$. Finally, it must be true that $[L(\mathrm{w}_1+a)-L(\mathrm{w}_1)] < \alpha\lambda[L(2\mathrm{w}_1)-L(\mathrm{w}_1)]$.

Region 4: From (22) the following must be true in equilibrium

$$\frac{1+\mu}{\beta} = (1-\lambda)[\alpha\lambda L(2\mathrm{w}_1) + (1-\alpha\lambda)L(\mathrm{w}_1)] + \lambda.$$

By the implicit function theorem, the following two must hold true.

$$\frac{d\mathrm{w}_1}{d\lambda} = \frac{[1-L(\mathrm{w}_1)] + \alpha(1-\lambda)[L(2\mathrm{w}_1) - L(\mathrm{w}_1)] - \alpha\lambda[L(2\mathrm{w}_1) - L(\mathrm{w}_1)]}{-(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1) + (1-\alpha\lambda)L'(\mathrm{w}_1)]}, \tag{a.8}$$
$$\frac{d\mathrm{w}_1}{d\alpha} = -\frac{(1-\lambda)[\lambda L(2\mathrm{w}_1) - \lambda L(\mathrm{w}_1)]}{(1-\lambda)[\alpha\lambda 2L'(2\mathrm{w}_1) + (1-\alpha\lambda)L'(\mathrm{w}_1)]} < .$$

where (a.8) holds since $1 - L(\mathrm{w}_1) < \alpha\lambda[L(2\mathrm{w}_1) - L(\mathrm{w}_1)]$.

□

Proof. Proof of Proposition 3

First, we show why $\partial\Pi/\partial\lambda > 0$ in Region 1 under the log-concave utility case. Since we are in Region 1, the following must be true.

$$\Pi'(\lambda) = -\alpha[u(q^*) - q^* - \{u(q_1^n) - q_1^n\}] - (1-\alpha\lambda)[u'(q_1^n) - 1)]\frac{dq_1^n}{d\lambda}, \tag{a.9}$$
$$\frac{dq_1^n}{d\lambda} = \frac{dq_1^n}{d\mathrm{w}_1}\frac{d\mathrm{w}_1}{d\lambda}, \text{ and } \frac{dq_1^n}{d\mathrm{w}_1} = \frac{1}{z'},$$

where the last equality comes from $\mathrm{w}_1 = z(q_1^n)$. Note that we chose to ignore $\beta(1-\eta)$ since it won't affect the result. Combining the above equations with (a.5), one can get

$$\begin{aligned}
\frac{dq_1^n}{d\lambda} &= \frac{1}{z'(q_1^n)}\frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{u'(q_1^n)/z'(q_1^n) - 1}{L'(\mathrm{w}_1)} = \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{u'(q_1^n) - z'(q_1^n)}{[z'(q_1^n)]^2 L'(\mathrm{w}_1)} \\
&= \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{u'(q_1^n) - z'(q_1^n)}{[u''z' - u'z''][1/z']} \\
&= \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{(u'-z')z'}{z'u'' - z''u'} = \frac{1-\alpha\lambda+\alpha(1-\lambda)}{(1-\lambda)(1-\alpha\lambda)}\frac{(u'-1)[(1-\eta)u' + \eta]}{u''}.
\end{aligned} \tag{a.10}$$

Note that the equality in the second line above comes from $L'(\mathrm{w}_1) = (u''z' - u'z'')/((z')^2)(1/(z'))$, and the 3rd one is from $z' = (1-\eta)u' + \eta,$ and $z'' = (1-\eta)u''$. By combining (a.9) and (a.10), one finally gets

$$\Pi'(\lambda) = \alpha[u(q_1^n) - q_1^n] - \alpha[u(q^*) - q^*] - \frac{1-\alpha\lambda+\alpha(1-\lambda)}{1-\lambda}\frac{[u'(q_1^n) - 1]^2[(1-\eta)u'(q_1^n) + \eta]}{u''(q_1^n)},$$

$$H(q) \equiv \alpha[u(q) - q] - \frac{1-\alpha\lambda+\alpha(1-\lambda)}{1-\lambda}\frac{[u'(q) - 1]^2\overbrace{[(1-\eta)u'(q) + \eta]}^{z'(q)}}{u''(q_1^n)}.$$

Then $\Pi'(\lambda) = H(q_1^n) - H(q^*)$. Hence, it suffices to show that H is decreasing in q for the proof of

$\Pi'(\lambda) > 0$. After some algebra, one can show

$$H'(q) = \frac{u'-1}{(1-\lambda)[u'']^2}\Big\{\alpha(1-\lambda)(u'')^2(1-z') - (1-\alpha\lambda)(u'')^2 z' \\ - [1-\alpha\lambda+\alpha(1-\lambda)][z'\{(u'')^2 - u'u'''\} + (u'-1)(1-\eta)(u'')^2 + z'u''']\Big\} < 0. \tag{a.11}$$

Inequality (a.11) holds true as long as $(u'')^2 > u'u'''$. This completes the proof.

Next, we prove $\partial\Pi/\partial\alpha > 0$ in Region 1 under any utility functional form. Ignoring $\beta(1-\eta)$, $\partial\Pi/\partial\alpha$ in Region 1 is defined as below.

$$\partial\Pi/\partial\alpha = -\lambda\left[u(q^*) - q^* - (u(q_1^n) - q_1^n)\right] - (1-\alpha\lambda)\left[u'(q_1^n) - 1\right]\frac{dq_1^n}{d\alpha}.$$

Let $G(q) \equiv \lambda(u(q_1^n) - q_1^n) - (1-\alpha\lambda)\left[u'(q_1^n) - 1\right] dq_1^n/d\alpha$. Then, $\partial\Pi/\partial\alpha = G(q) - G(q^*)$. Hence, if $G'(q) < 0$ then $\partial\Pi/\partial\alpha > 0$. One needs to show $\partial\Pi/\partial\alpha > 0$ only for this proof. Similar to (a.10), one can derive the following.

$$\begin{aligned}\frac{dq_1^n}{d\alpha} = \frac{dq_1^n}{d\mathrm{w}_1}\frac{d\mathrm{w}_1}{d\alpha} &= \frac{1}{z'(q_1^n)}\frac{-(1-\lambda)\lambda[1-L(\mathrm{w}_1)]}{(1-\lambda)(1-\alpha\lambda)L'(\mathrm{w}_1)} \\ &= \frac{-\lambda[z'-u']}{(1-\alpha\lambda)[u''z'-u'z'']}z' \\ &= \frac{\lambda}{1-\alpha\lambda}\frac{z'(u'-1)}{u''},\end{aligned} \tag{a.12}$$

where the second line in (a.12) follows from $L'(\mathrm{w}_1) = (u''z' - u'z'')/(z')^2(dq_1^n/d\mathrm{w}_1)$, and the third line is from $z'' = u''(1-\eta)$ and $z' = (1-\eta)u' + \eta$. After some algebra along with (a.12), the following can be derived.

$$\begin{aligned}G'(q) &= \frac{u'-1}{1-\alpha\lambda}\left\{\lambda(1-\alpha\lambda) - \lambda(1-\alpha\lambda)u' + \eta(1-\alpha\lambda)\lambda[u'-1] - (1-\alpha\lambda)\lambda\eta\right\} \\ &= \frac{u'-1}{1-\alpha\lambda}\left\{\lambda(1-\alpha\lambda)\left(1+\eta[u'-1]\right) - \lambda(1-\alpha\lambda)\left(u'+\eta\right)\right\} < 0,\end{aligned}$$

where the inequality is true since $1-\eta-\eta < u'(1-\eta)$. This completes the proof.

□