Dynamic Asset Allocation under Disappointment Aversion Preferences*

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Abstract

We study the impact of Disappointment Aversion (DA) preferences of Gul (1991) over portfolio allocation in a dynamic setting by updating the certainty equivalent endogenously. In our paper, particular emphasis is given to the effects of return predictability and parameter uncertainty on long-term portfolio allocation. By calibrating relevant data generating processes, we find that horizon effects persist, even in the case of i.i.d. returns, with stocks appearing progressively more attractive at longer horizons. These effects intensify when investors become more disappointment averse, thus confirming the role of DA as an explanatory factor underlying the non-participation puzzle in the stock market.
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1 Introduction

The debate on the importance of asset allocation decisions for individuals has been progressively rising in prominence in public discourse, with policy makers growing more preoccupied with its implications for various domains of individuals' welfare (e.g., retirement planning; see Thaler and Sunstein, 2008). Key to policy makers' concerns is the fact that investors are susceptible to psychological forces capable of biasing the selection of asset classes they allocate their wealth to, leading to potentially sub optimal choices of asset mix.[1] Prospect theory (Kahneman and Tversky, 1979) offers an excellent example of how such forces can impact asset allocation, by showcasing how the interplay of various biases/heuristics (including anchoring, framing and mental accounting) prompts individuals to evaluate the performance of their investments on a relative basis, vis-a-vis some actual, historical reference point. This leads to the encoding of performance into gains and losses, to which investors have been shown to respond asymmetrically, appearing risk-seeking in the domain of losses (by holding onto losing stocks) in order to avoid realizing them (loss aversion) and risk-averse in the domain of gains (by selling their winning stocks quickly – while they are still winning). However, comparing actual past and contemporaneous values in assessing an investment's performance represents only one possibility of relative performance evaluation; in reality, individuals engage in a risky investment in anticipation of positive returns, in which case what also matters to them is not only whether the investment's realized return in the future is positive or not, but also whether it is above or below their initial expectations. A below-expectations performance of an investment can foment disappointment (Fielding and Stracca, 2007) whose incorporation in an investor's learning pro-

[1] Investors may choose, e.g., to invest more aggressively in equities motivated by representativeness, if equities have recently performed well, thus potentially leading to trend-chasing and herding in specific stocks and/or sectors (Choi and Sias, 2009). Another possibility is that, when presented with a finite number, let n, of investment options, investors choose to allocate their assets equally to each one of them (i.e., the well-known $1/n$ rule), a behaviour best known as "naive diversification" (Benartzi and Thaler, 2001).

cess can introduce disappointment aversion tendencies in her trading behaviour. Although disappointment aversion has been shown to be important in asset allocation decisions in a single-period setting (Ang et al., 2005), no research to date has investigated its effect over asset allocation in a dynamic setting (admittedly, a more accurate reflection of investment reality, compared to its static/single-period counterpart) and it is this issue that the present study aims at addressing.

Our research is primarily motivated by extant evidence, according to which investors do not strictly adhere to the assumptions of expected utility theory in their decisions[2], being prone to viewing choices in a biased fashion instead, often under the influence of emotions and cognitive biases (such as mental accounting and framing effects).[3] Several theoretical propositions (Handa, 1977; Chew and MacCrimmon, 1979; Quiggin, 1982; Fishburn, 1983; Tversky and Kahneman, 1992) depart from the expected utility framework to reflect more accurately investors' decision making under risk, transforming probabilities into decision weights through non-linear probability functions. Prospect theory (PT, hereafter), in particular, has proved particularly successful in capturing frequently encountered traits of investors' behaviour (Kahneman and Tversky, 1979; Berkelaar et al., 2004; Gomes, 2005; Barberis and Huang, 2008; Dimmock and Kouwenberg, 2010; Bernard and Ghossoub, 2010). According to its context, investors are assumed to evaluate the performance of their investments by anchoring on some historical reference point, engaging in the computation of gains and losses relative to that point. Investors also respond asymmetrically to gains versus losses – courtesy of loss aversion – by exhibiting greater sensitivity to losses compared to gains, something further re-

[2]In practice, for example, the independence axiom is frequently violated, with the Allais paradox representing the most famous evidence of the latter. Other notable violations of the expected utility framework are observed in Ellsberg's paradox and the St. Petersburg paradox. For more on those, see Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979) and Andreoni and Sprenger (2010).

[3]In the case of mental accounting, investors holding a portfolio of stocks may treat the performance of each stock in isolation, instead of viewing it as part of the portfolio (Barberis et al., 2001). Framing, on the other hand, can lead investors to choose an option not because it is optimal, but rather because it appears attractive on the background of less attractive alternatives. For a more detailed discussion of the above, see Kahneman et al. (2011).

flected through the PT value function, which grows steeper in the loss region. As a result, they grow more risk-seeking when in the domain of losses (they hold onto their loser stocks in hope of a price rebound) and more risk-averse in the domain of gains (they sell their winner stocks in order to realize their profits while they still exist), thus ending up selling their winning assets more quickly compared to their losing ones.[4]

One of the derivatives of PT is Disappointment Aversion (DA, hereafter) theory, formally introduced by Gul (1991).[5] DA theory extends the expected utility theory by relaxing the independence axiom, whilst also retaining the basic features of prospect theory (asymmetric preferences; reference dependence; diminishing sensitivity; and probability weighting). Moreover, it provides us with better understanding in the way the certainty equivalent is chosen and updated. Certainty equivalent represents the certain level of wealth W that generates the same level of utility as a portfolio composition which yields a (non-certain) wealth level W and in a DA context serves as a reference point of investor's wealth against which gains and losses are compared. In PT these points are set exogenously and are usually equal to the current wealth (the status quo), while in DA theory they are updated in an endogenous way. In a later work, Tversky and Kahneman (1992) presented a more tractable way to define the certainty equivalent by setting it equal to the midpoint of a set of ordered outcomes with the smallest representing the "lowest accepted value" and the largest the "highest rejected" one. This setup, although closer to

[4]Empirical evidence (Odean, 1998; Grinblatt and Keloharju, 2001; Locke and Mann, 2001; Shapira and Venezia, 2001; Wermers, 2003; Haigh and List, 2005; Jin and Scherbina, 2010) suggests that this pattern permeates both retail and institutional investors' behaviour internationally, yet leads to sub optimal performance. The latter has been ascribed to the effect of short term momentum (Jegadeesh and Titman, 1993, 2001) in stock returns, according to which recent winners (losers) will continue outperforming (underperforming) in the near future; this, in turn, suggests that investors in the prospect theory setting should be keeping their winners (instead of selling them quickly) and selling their losers (instead of keeping them). A potential explanation for the momentum effect can be given in the context of a realization utility model (Barberis and Xiong, 2012), where the selling of well-performing stocks to realize gains is attributed to the additional utility derived by real instead of paper profits.

[5]Although this framework is recognized as the DA theory one, Bell (1985) first studied the disappointment effect arising from the discrepancy between an agent's prior expectations and realized outcomes.

the way the reference points are updated in DA theory, does not suggest an entirely endogenous way to that end, and thus we believe that the DA framework in Gul (1991) offers a better understanding on this technicality (Ang et al., 2005).

Overall, empirical applications of DA theory have been rather limited to date, a fact attributed by Abdellaoui and Bleichrodt (2007) to it lacking a method of formally extracting the DA coefficient. To that end, Abdellaoui and Bleichrodt (2007) proposed the trade-off method, which first derives the underlying utility function and then, based on that, extracts the DA coefficient. DA theory has been mainly used in asset pricing settings (Routledge and Zin, 2010; Bonomo et al., 2011) where a slightly altered version of the original DA theory is used. More specifically, these studies consider a generalized version of Gul (1991)'s framework, extending the DA utility by an additional term and a new coefficient on top of the DA coefficient - which shifts the disappointment region for an outcome. In these setups, an outcome signals "disappointment" only when it lies sufficiently below the certainty equivalent, as determined by the additional coefficient and the DA parameter. In asset allocation setups, Dahlquist et al. (2017) employed DA preferences to derive analytical expressions for measures such as the effective risk aversion, when studying higher moments of return distributions.

With regards to portfolio choices, Ang et al. (2005) first addressed this issue in a single-period setting, with DA-utility investors allocating their wealth between a risk-free security and a risky asset, while also assessing the robustness of their results when accounting for loss aversion. Ang et al. (2005) find that the incorporation of DA leads to more plausible portfolio compositions with smaller proportion of wealth allocated to the risky asset, providing also a reasonable explanation for the observed equity premium – non-participation puzzle. As a result, DA is highly relevant to an individual's decision making and should be taken into consideration in asset allocation. Motivated by Ang et al. (2005), the present paper provides a thorough investigation of the impact of the DA theory on the problem of asset allocation in a dynamic, long-horizon setting.

Studying DA in a dynamic framework allows us the opportunity to gauge whether investors allocate different weights to their portfolio assets throughout the investment horizon, thus rendering our study more realistic, compared, for example, to a static or a single-period one.[6] Our study first revisits and completes the single-period version of DA utility by Ang et al. (2005), before developing a discrete-time dynamic framework in partial equilibrium[7] which allows for sequential investing and re-allocation of the available wealth, to investigate how DA preferences affect the decision making of individuals who seek to maximize their terminal utility of wealth. Investors choose portfolio allocations for two investable assets, a risk-free bond and a risky asset. Since it is impossible to, a priori, assert investors' beliefs as per the generation process of returns in the market we follow the relevant literature by allowing for two possible Data Generating Processes (DGPs). The first one assumes returns are independent and identically distributed (i.i.d., i.e., the observation at time t does not relate to the previous one at time $t-1$); the second one assumes a VAR-form, which uses the dividend price ratio (i.e., the dividend yield) as its predictor variable.

An issue arising with the above two DGPs pertains to the uncertainty inherent in their estimated parameters and how it may be treated by investors. To accommodate uncertainty, we consider a Bayesian framework following the relevant literature on Bayesian portfolio analysis (see Bawa et al., 1979; Kandel and Stambaugh, 1996; Barberis, 2000; Avramov and Zhou, 2010; Kacperczyk and Damien, 2011, among others; see also footnote 8). To assess the impact of parameter uncertainty, we compare portfolio allocations to the risky asset between cases where parameter uncertainty is ignored and others where it is considered. The difference between an

[6]Dynamic portfolio allocation in general has, overall, been widely studied, both in discrete and continuous time (Campbell and Viceira, 2002; Brandt et al., 2005; Aït-Sahalia et al., 2009).

[7]Our approach entails an exogenous price setting, which makes this study a partial equilibrium one. Modelling the cash flows can lead to an endogenous price setting where equilibrium asset prices are attained and markets clear. For an example of an equilibrium study, see Lynch (2000).

investor who ignores parameter uncertainty and one who incorporates it in an asset allocation problem lies in the way they use the given DGP to generate future return paths. The former considers the values of the model's parameters as true, while the latter recognizes the underlying risk in their estimation. In a dynamic setting the incorporation of new information about equity returns alters the estimated parameter sets at different horizons, introducing uncertainty in model parameters which is modelled via their posterior distribution after considering the newly generated data. Analytically, this is performed by integrating out the variance in the posterior distribution of the parameters, constructing a distribution conditional on observed data and not on the model parameters. To capture the difference between the two approaches, we perform numerical experiments with and without parameter uncertainty, assuming both i.i.d. and predictable risky asset returns. Overall, our results indicate that equity allocation drops with the investment horizon as a result of the higher volatility of the risky asset, which in turn stems from the additional uncertainty around the true values of the model parameters.[8]

Our study presents evidence strongly supporting the role of DA in defining equity participation (and non-participation) regions. We show, both mathematically and empirically, that, for every portfolio allocation and level of expected equity premium, there is a critical value of DA below which it is optimal for a DA investor to hold zero units of the risky asset (stock). Perhaps more interestingly, we find that DA investors tend to allocate significantly less to equity compared to those with isoelastic (power) utility. DA appears to be powerful at every horizon as we find that a small drop of the DA coefficient leads to a significant decrease in

[8]Evidence from recent research confirms the importance of predictability and parameter uncertainty for portfolio choices. Branger et al. (2013) and DeMiguel et al. (2015) examine the construction of optimal portfolios under uncertainty about expected asset returns and find that parameter uncertainty is highly relevant to portfolio choice. Chen et al. (2014) study the dynamic portfolio choice problem when investors face uncertainty about the model specification, incorporating learning as well to construct strategies which depart from the Bayesian approach. Hoevenaars et al. (2014) test the impact of different uninformative priors on both short and long-term equity allocations, while Johannes et al. (2014) investigate the impact of predictability and parameter uncertainty in an expected utility framework focusing mainly on the impact of volatility on the portfolio choice problem.

equity holdings in the case of an investor who accounts for predictability in stock returns. We also observe the rise of differential horizon effects when parameter uncertainty is either ignored or incorporated for investors who either use the i.i.d. return generating process or account for return predictability. A moderately risk averse, buy-and-hold investor will allocate a large part of her wealth to the risky asset for a sufficiently long horizon when parameter uncertainty is ignored, taking advantage of the lower per-period volatility of the risky asset's returns which, in turn, decreases the cumulative volatility she experiences over the investment horizon. On the other hand, a DA investor who follows a dynamic strategy and assumes predictability in returns will decrease their allocation to the risky asset at shorter horizons. Again, the impact of the DA coefficient is drastic, as the more DA the investor becomes the less the portfolio weight she assigns to equity. We also notice that, when parameter uncertainty is incorporated, equity allocation at shorter horizons is significantly lower to that at longer ones.

Our paper produces a series of original contributions to the extant literature on portfolio choice under uncertainty. First, we extend the study of portfolio choice for investors with DA utility by providing optimal participation conditions and non-participation regions both for static (buy-and-hold) and dynamic allocations. Second, we revisit and extend the study of the portfolio choice problem for a long-term buy-and-hold investor under return predictability and parameter uncertainty. Although our study primarily focuses on dynamic portfolio choice, revisiting the buy-and-hold asset allocation problem for very long investment horizons (up to 40 years) reveals a number of important implications as regards the different investment behaviour of a long vs a short - term DA investor, which – to our knowledge – have not attracted attention so far. Third, we demonstrate how the incorporation of predictability in asset returns affects portfolio weights at different horizons for a dynamic investor and how this can give rise to horizon effects, in the sense that investors change their portfolios' compositions taking into account the variability in investment opportunities. Fourth, we complete our study constructing a Bayesian framework which incorporates both predictability and parameter uncertainty to investigate how each of the two properties affect portfolio compositions in a DA context. Here, the choice of the risky asset return generator is crucial; for example, the impact of parameter uncertainty on a dynamic strategy where returns are i.i.d. is not as powerful as in the case where predictability is considered, leading to significantly different portfolio allocations.

Our results should be of particular interest to policy makers, as they indicate that DA, conditional on its magnitude, tacitly fosters limited-to-no participation in equity investing. To the extent that DA is likely to affect individual investors more (given their lower sophistication levels), financial literacy programmes could raise awareness of it, while at the same time train individuals to assess their investments from a longer-term perspective, irrespective of price movements in the short run (where the effects of DA are more likely to be felt). This, in turn, will help enhance the participation of retail investors in equity turnover (thus benefiting market liquidity), while also ensuring that those that do invest in equities are less likely to exit the market due to disappointment-related reasons. Our results are also relevant to finance practitioners, in particular brokers and financial advisors, who, by virtue of their profession, tend to engage on a regular basis with retail investors. For these practitioners, accounting for DA in their clients' risk profiling and overall day-to-day interactions would help inform considerably their professional practice, by permitting them additional insight into their clients' trading decisions. Such insight could allow them to educate their clients as per the role of DA in trading, thus helping them potentially improve on their trading decisions. From an academic perspective, our results further contribute to the debate on the equity premium puzzle, as they showcase that DA constitutes a plausible explanation underlying the relative reluctance of investors to hold equities. What is more, to the extent that disappointment stems from prior investment experience, our results also offer an alternative explanation to previously documented evidence (Seru et al., 2010; Strahilevitz et al., 2011) on the reluctance of investors to re-enter the market if they have exited it previously at a loss.

Our paper is structured as follows; Section 2 presents the DA framework for the single and multi-period case, along with the DA algorithm developed for the dynamic problem and the theoretical framework for the predictability and parameter uncertainty elements of our empirical design. In Section 3 we present and discuss a number of numerical examples, followed by concluding remarks in section 4. Finally, three appendices can be found in the end of this paper with all the technical details for the multi-period problem formulation, the algorithmic procedure and the incorporation of parameter uncertainty and predictability in the DA asset allocation context.

2 The disappointment aversion framework

The DA framework employed in this study is defined as in Ang et al. (2005):

$$U(\mu_W) = \frac{1}{K}\left(\int_{-\infty}^{\mu_W} U(W)dF(W) + A\int_{\mu_W}^{\infty} U(W)dF(W)\right), \tag{1}$$

where A is the coefficient of DA, bounded between zero and one (i.e., $0 < A \leq 1$), $U(\cdot)$ is the constant relative risk aversion (CRRA) utility function defined by $U(W) = W^{1-\gamma}/(1-\gamma)$, μ_W is the certainty equivalent of wealth, $F(\cdot)$ is the cumulative distribution function for wealth and K is a scalar equal to $P(W \leq \mu_W) + AP(W > \mu_W)$. Assume two assets, one risky asset and one risk-free asset, whose continuously compounded returns are denoted by e^y and e^r, respectively. Then, the investor's wealth is defined as $W = \alpha X + e^r$ where α is the investment in the risky asset as a percentage of the total investment (i.e., the weight of the risky asset), $X = e^y - e^r$ is the excess risky asset's return and the initial wealth is set to one, since the optimization problem is homogeneous in wealth under the CRRA utility function. When a DA investor allocates her wealth into assets in order to maximize the DA utility for a single period, the static optimization problem is,

$$\max_{\alpha} U(\mu_W). \tag{2}$$

The above constitutes the asset allocation problem under the assumption of DA utility in a single period setting that has been extensively investigated in Ang et al. (2005).

2.1 Dynamic allocation with disappointment aversion utility

A dynamic optimization problem with DA utility in a multiple-period setting is considerably more complicated than in a static one, because at every horizon the optimization routine should take into account the investment opportunity set for the whole remaining investment period (as opposed to a one-period forward looking myopic strategy), while the certainty equivalent of wealth is itself a function of each step's optimal decision. The complexity of the optimization problem further increases by considering a DGP with predictability, which, as a result, leads to stochastic investment opportunity sets. We begin by first analysing a conventional utility function defined over wealth $U(W)$ and then move to the dynamic asset allocation with DA utility.

2.1.1 Dynamic Asset Allocation with Utility of Wealth U(W)

Assume the following dynamic asset allocation problem in discrete time, where an agent aims to maximize the expected utility of the end-of-period wealth W_T as follows:

$$\max_{\alpha_0,\alpha_1,\ldots,\alpha_{T-1}} \mathbb{E}_0[U(W_T)], \tag{3}$$

where $\alpha_0, \alpha_1, ..., \alpha_{T-1}$ are the investment proportions of the risky asset at times $t = 0, 1, \ldots, T-1$, respectively, and $U(W) = W^{1-\gamma}/1-\gamma$. In this problem, the investor allocates her wealth at time $t = 0$ for T periods, at $t = 1$ for $T-1$ periods and so on, until she reaches time $t = T-1$ where she invests for a single period.[9] Wealth W_{t+1} is defined as $W_{t+1} = W_t R_{t+1}(\alpha_t)$, where $R_{t+1}(\alpha_t)$ and α_t represent the total portfolio return over the period t to $t+1$ and the investment weight on the risky asset at time t, respectively. At time t when the investor seeks to allocate optimally her available wealth between the risky and the riskless asset in order to maximize her expected utility, the optimization problem becomes

$$\max_{\alpha_t} \mathbb{E}_t[U(W_{t+1}Q^*_{t+1,T})] \tag{4}$$

where $Q^*_{t+1,T} = R_T(\alpha^*_{T-1})R_{T-1}(\alpha^*_{T-2}) \cdots R_{t+2}(\alpha^*_{t+1})$ represents the aggregate return from time $t+1$ to T that maximizes the investor's expected utility. Using dynamic programming, we solve the problem at time $t = T-1$ for the asset allocation decision for the period $T-1$ to T. Continuing recursively, we solve the asset allocation sub-problem at time $T-2$ using the solution of the problem at $T-1$, until we reach time t. This procedure derives a final solution for the portfolio allocation to the risky asset $\alpha_t, \alpha_{t+1}, \ldots, \alpha_{T-1}$ that will be optimal as guaranteed by the principle of optimality in dynamic programming.[10] We apply the backward induction to the conventional CRRA utility function. By plugging in the power utility function, Eq. (4) takes the form of

$$\max_{\alpha_t} \mathbb{E}_t\left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma}(Q^*_{t+1,T})^{1-\gamma}\right]. \tag{5}$$

Backward induction suggests that $Q_{t+1,T}$ is optimal, because it represents the optimal investment decision between times $t+1$ and T, that maximizes the expected

[9] This problem mimics the optimization problem that pension fund managers face over multiple periods.

[10] See Bertsekas (1995) for more details on that.

utility. Using dynamic programming we calculate the optimal investment proportions of the risky asset at every time-step of the investment period as:

$$\alpha_t^* = \arg\max_{\alpha_t} \mathbb{E}_t\Big[W_{t+1}^{1-\gamma}(Q_{t+1,T}^*)^{1-\gamma}\Big]. \quad (6)$$

2.1.2 Dynamic Asset Allocation with DA Utility

DA utility incorporates CRRA preferences as a special case when $A = 1$, but the dynamic extension of the single-period problem for DA utility is far more complicated, because of the so-called "curse of dimensionality": the number of state variables increases exponentially with time.[11] We begin by first formulating the dynamic optimization problem between t and T.

Proposition 1 *For given $Q_{t+1,T}^* = R_T(\alpha_{T-1}^*)R_{T-1}(\alpha_{T-2}^*)\cdots R_{t+2}(\alpha_{t+1}^*)$, the DA utility function for the dynamic asset allocation problem is given by*

$$U(\mu_t) = \frac{1}{K_t}\Bigg[\mathbb{E}_t\Big(U(W_{t+1}Q_{t+1,T}^*)\mathbf{1}_{W_{t+1}Q_{t+1,T}^*\leq\mu_t}\Big) + A\mathbb{E}_t\Big(U(W_{t+1}Q_{t+1,T}^*)\mathbf{1}_{W_{t+1}Q_{t+1,T}^*>\mu_t}\Big)\Bigg], \quad (7)$$

where $W_{t+1}Q_{t+1,T}^* = W_T$, according to the recursive definition of wealth. The FOC for the optimization of the utility of the certainty equivalent return is given by

$$\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q_{t+1,T}^*R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\leq\mu_t}\right) + A\mathbb{E}_t\left(\frac{dU(W_T)}{dW}Q_{t+1,T}^*R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\right) = 0, \quad (8)$$

where $X_{t+1} = e^{y_{t+1}} - e^{r_t}$ is the excess return of the risky asset over the riskless

[11] As the state variables take a number of different values at each horizon, the state-space increases exponentially with time with every iteration of the algorithm. For example, a T-period problem with a state-variable with s states produces s^T possible combinations. While from an analytical perspective this is not a big obstacle (as the problem can still be mathematically formulated), computation-wise, the exponential increment of the state-space renders the use of algorithmic processes problematic.

asset.

Proof. See Appendix C.1

The main drawback with Proposition 1 is that the recursive optimization increases the state space in $Q_{t+1,T}$ exponentially in order to take into account all the possible states for the return of the risky asset between times $t+1$ and T. To overcome the "curse of dimensionality" problem we elaborate on the approach proposed in Epstein and Zin (1989) and Ang et al. (2005), by making the assumption that future uncertainty about the risky asset's returns is captured by the certainty equivalent. Under this approach, instead of carrying backwards all the possible states for the equity return at each horizon, we pay attention to only one variable, next period's certainty equivalent, keeping the dimension of the state-space to the minimum possible. Let μ_t represent the certainty equivalent return for the utility at time $t+1$ with the optimal asset allocation:

$$\max_{\alpha_t} \mathbb{E}(U(W_{t+1})) = \max_{\alpha_t} U(W_t \mu_t(\alpha_t)). \tag{9}$$

Then, we obtain the following result:

Proposition 2 *The utility of the certainty equivalent return at time $0 \leq t < T-1$ is as follows:*

$$\begin{aligned} U(\mu_t) = \frac{1}{K_t} \Bigg[\mathbb{E}_t \Bigg(U(R_{t+1}(\alpha_t) W_t \prod_{i=t+1}^{T-1} \mu_i^*) \mathbf{1}_{\{R_{t+1}(\alpha_t) \leq \xi_t\}} \Bigg) \\ + A \mathbb{E}_t \Bigg(U(R_{t+1}(\alpha_t) W_t \prod_{i=t+1}^{T-1} \mu_i^*) \mathbf{1}_{\{R_{t+1}(\alpha_t) > \xi_t\}} \Bigg) \Bigg]. \end{aligned} \tag{10}$$

The value of $U(\mu_t)$ for the boundary condition $t = T-1$ is given by:

$$\begin{aligned} U(\mu_{T-1}) = \frac{1}{K_{T-1}} \Big[\mathbb{E}_{T-1} \big(U(R_T(\alpha_{T-1}) W_{T-1}) \mathbf{1}_{\{R_T(\alpha_{T-1}) \leq \mu_{T-1}\}} \big) \\ + A \mathbb{E}_{T-1} \big(U(R_T(\alpha_{T-1}) W_{T-1}) \mathbf{1}_{\{R_T(\alpha_{T-1}) > \mu_{T-1}\}} \big) \Big], \end{aligned} \tag{11}$$

and the FOC for the optimization of the utility of the certainty equivalent return is given by:

$$\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\xi_t\}}\right)+A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}}\right)=0, \tag{12}$$

where $\xi_t = \frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}$, *with* μ^**'s the optimal certainty equivalents between* $t+1$ *and* $T-1$*.*

Proof. See Appendix C.2.

Remark *Notice that eventually* W_t *will not be part of the expressions for* $U(\mu_t)$ *in Eqs. (10) and (11) as moving backwards in time we will have* $W_t = W_0\prod_{i=1}^{t} R_i(\alpha_{i-1})$ *where all the uncertainty about* $R_n(\alpha_{n-1})$ *will be captured by the certainty equivalent return* μ^*_n *and* W_0 *is set to one given that the optimization problem is homogeneous in wealth.*

We also notice that, the investor's gains or losses at time $t+1$ are now calculated with respect to ξ_t, that is the certainty equivalent at time t with respect to the optimal certainty equivalent from the period from $t+1$ to T. By substituting portfolio returns with the corresponding certainty equivalent, the state-space comprises a constant number of states which remains unchanged with time. As an example of the advantage of using the certainty equivalent, we can present the FOC in Eq. (12) for power utility as follows:

$$\mathbb{E}_t\big(R_{t+1}^{-\gamma}(\alpha_t)X_{t+1}\mathbf{1}_{R_{t+1(\alpha_t)\leq\xi_t}}\big)+A\mathbb{E}_t\big(R_{t+1}^{-\gamma}(\alpha_t)X_{t+1}\mathbf{1}_{R_{t+1(\alpha_t)>\xi_t}}\big)=0. \tag{13}$$

To find the optimal numerical values for μ_t and α_t, we adopt a Gaussian quadrature scheme (see Davis and Rabinowitz, 2007, for an in–depth review of numerical integration methods) as in Balduzzi and Lynch (1999) and Campbell and Viceira (1999) to track the states $\{R^s_{t+1}(\alpha_t)\}_{s=1}^{N}\big(\prod_{i=t+1}^{T-1}\mu^*_i\big)$, where N is the number of quadrature states for the equity return.[12] Next, we solve the discretized expression of Eq. (10) (adjusted for power utility) in parallel with the FOC for the DA maximization problem in Eq. (13) as in the static single-period case, incorporating now recursively the calculations from periods $T-1$ to $t+1$. Details on the discretization of the DA allocation problem and its solution can be found in Appendix B.

2.2 Non-participation under DA utility

The case for non-participation in risky assets has been the subject of considerable research to date. Motivated by mental accounting (which assumes the non-fungibility of monetary resources allocated to each asset; see e.g., Thaler and Sunstein, 2008), narrow framing (Barberis and Huang, 2008) can prompt investors to perceive high-volatility assets as "risky" in isolation, without assessing their contribution to the risk-return profile of a portfolio. Non-participation can also be promoted by the omission bias (Ritov and Baron, 1999), whereby omissions are favored over equivalent commissions because, unlike omissions (e.g., not investing in stocks), commissions (investing in stocks) involve commitment to a course of action, thus entailing the possibility of a loss. Other alternative explanations proposed to account for non-participation include familiarity bias (choosing more over less familiar assets, believing the latter to be riskier; Huberman, 2001; Massa and Simonov, 2006), recognition bias (preferring more over less recognizable assets; Boyd, 2001) and limited cognition (if investors view risk-diversification as a decision of enhanced complexity; Hirshleifer, 2008).

[12] Instead of quadrature–based methods, Monte–Carlo simulations or even regression–based methods as in Brandt et al. (2005) can be used to calculate the expectations in Eq. (13). In practice however, the quadrature method offers sufficient accuracy and greater computational speed compared to the alternatives.

2.2.1 Single-period

Although under CRRA preferences it is always optimal to hold positive portfolio weights for risky assets when the expected excess return is positive ($\mathbb{E}(X) > 0$), this is not always the case with DA utility preferences. Under DA preferences there can be cases where it is optimal to refrain from holding risky assets even if the expected excess return is positive. This non-participation region in the following theorem shows that it is not optimal to hold risky assets whenever the DA coefficient lies below a critical value (A^*).

Theorem 1 *Let $\mu = \mu_W(A, \alpha)$, with*

- $\mu(A, :) \in \mathbf{C}^1, \forall A \in [0, 1]$
- $\frac{d\mu(A,0)}{d\alpha} = \xi(A) \leq 0, \forall A \in [0, 1]$[13]
- $\mathbb{E}(X) > 0$ *and* $\mathbb{E}(X\mathbf{1}_{W \geq \xi(A)}) > 0$, *where* $X = e^y - e^r$ *is the return of the risky asset in excess of the risk–free rate.*

Then, setting

$$A^* = \frac{\mathbb{E}(X\mathbf{1}_{W \geq \xi(A)})}{\mathbb{E}(X\mathbf{1}_{W < \xi(A)})}, \tag{14}$$

we have the following:

1. *For every $A \leq A^*$, $\alpha^* = 0$,*
2. *For every $A > A^*$, $\alpha^* > 0$,*

where α^* is the optimal investment proportion in the risky asset which maximizes $\mu(A, \alpha)$ for a given A. A^* is independent of the risk aversion parameter γ.

Proof. See appendix C.3

Intuitively this theorem can be presented in the following way: focusing on the DA

[13]Positive risk premium when the end-of-period wealth exceeds the negative impact of the decrease in the certainty equivalent as the investment proportion of the risky asset increases. Suppose that the expected return of the risky asset is zero. The certainty equivalent decreases when the proportion of the risky asset increases. This occurs because for $\alpha < 0$ negative excess return states have higher wealth than R and hence are downweighted (Ang et al., 2005).

coefficient A, we find that, as it decreases, investors allocate less wealth to the risky asset regardless of their level of risk aversion. Given that the utility of wealth is a continuous function within the domain of A, there should be a level of A, let A^*, at which the optimal portfolio allocation to the risky asset, α^* equals zero. This result obtains independently of the level of risk aversion γ. Recalling the condition $d\mu(A,0)/d\alpha \leq 0$, we see that a further decrease in the portfolio weight allocated to the risky asset α^* (e.g., due to short-selling the risky asset) will result in a higher certainty equivalent. When investment in the risky asset is zero, an increase in the investment in the risky asset decreases the certainty equivalent. This is intuitively correct since by increasing the portfolio allocation to equities to a non-zero level we become more willing to take on an amount of risk instead of holding only the risk-free security. This consequently implies that the monetary amount which can keep us away from buying stocks should be now lower. Subsequently, the following relationship will prevail:

$$W = \alpha^* X + R > R,$$

for $\alpha^* < 0$ and negative states ($X < 0$) of the excess equity return. Therefore, the optimal allocation for this critical level of the DA coefficient, A^* is zero and $\alpha = \alpha^* = 0$.

[Figure 1 about here.]

2.2.2 Multi-period

To calculate A^* at different horizons, we perform a number of Monte-Carlo (MC) experiments for the buy-and-hold and dynamic allocation problems, where we simulate several asset return trajectories under the i.i.d. assumption and using the DGP with predictability to estimate the excess return and the corresponding return volatility. Then, using a binary search algorithm we are able to extract the critical DA coefficient A^* (which results in allocating zero wealth to equity) for each problem.[14] A typical binary search algorithm for our problem proceeds with discretizing the state space of A (i.e., assuming discrete points on the interval $(0, 1]$) and subsequently performing sequential searches for the target value of A (i.e., the one which makes $\alpha = 0$). This is done by comparing the target value to the middle element of the state space and cutting the state space in half with every iteration until the optimal value is detected (for implementation details of the binary search algorithm see sections 3.1 and 3.2 in Sedgewick and Wayne, 2011).

In the numerical experiments, we follow the simple assumption of normality, where risky asset returns follow an i.i.d. process, which we also relax by assuming predictable asset returns. To model the predictability of asset returns we use a vector autoregression (VAR) model which has been frequently used in the asset allocation literature in discrete time (see Barberis, 2000; Ang et al., 2005; Brandt et al., 2005; Hoevenaars et al., 2014, among others). In our VAR, after examining a number of candidate variables (see section 2.3.3 for more details) we choose the dividend price ratio $(d/p)_t$ as the main driver of next period's equity return.

In the MC simulations, quarterly data for the S&P 500 index and the 3-month T-bill are used as proxies for the equity returns and the risk-free rate, respectively. Our numerical experiments corroborate Theorem 1: risk-aversion, has no impact on the non-participation region for the DA coefficient.

[Figure 2 about here.]

The left graph of Fig. 2 plots the critical level of the coefficient of DA (A^*) across investment horizons for a buy-and-hold DA investor. The DA coefficient (A^*) is critical because a DA investor should not hold any units of the risky asset if her DA lies below A^*. A decreasing A^* within these setting results in larger market participation, as a lower A^* implies that the investor has to be more disappointment averse in order to refrain from holding the risky asset. For a longer than a five-year

[14] An alternative approach entails the calibration of a binary tree and the detection of the correct interval for A^* (Ang et al., 2005). In practice, both methods derive similar results for A^*.

period, a DA investor who follows a buy-and-hold strategy will hold risky assets regardless of the DGP assumed for equity returns.

The right graph of Fig. 2 reports critical levels of A^* for dynamic asset allocation strategies for various investment horizons ($T-t$, where t is the current horizon). In the case of i.i.d. returns (dashed line), the critical DA coefficient remains constant regardless of the investment horizon as a result of the invariable opportunity set. The solid line corresponds to predictable returns using the VAR to forecast next period's equity return as a function of the dividend price ratio. Contrary to the case of i.i.d. returns where A^* remains constant, participation increases at longer horizons.

2.3 Asset allocation with parameter uncertainty

The effects of parameter uncertainty on asset allocation can be investigated by allowing for uncertainty in the estimates of the parameters. At time t investors maximize the following utility function:

$$\max_{\alpha} \int_{-\infty}^{\infty} U(W_{t+n}) p(r_{t+n}|Y;\theta) dr_{t+n}, \tag{15}$$

where n is investor's horizon, $U(\cdot)$ is the utility of wealth and $p(r_{t+n}|Y;\theta)$ is the cumulative density function of the expected returns conditional on observed return data Y and the set of parameters θ (in our case the mean and variance of risky asset's return). The uncertainty arises with respect to θ, since these parameters become known after we reach the end of the investment horizon. One of the popular approaches in the literature to deal with the parameter uncertainty problem, is using a Bayesian framework that incorporates uncertainty in the parameters of θ. Integrating out θ in the prior distribution $p(r_{t+n}|Y;\theta)$, we end up with the posterior predictive distribution which updates the distribution parameters by embodying the new data. A DA investor now maximizes

$$\max_{\alpha}\left[\int_{W_{t+n}\le\mu_W} U(W_{t+n})p(r_{t+n}|Y)dr_{t+n} + A\int_{W_{t+n}>\mu_W} U(W_{t+n})p(r_{t+n}|Y)dr_{t+n}\right], \tag{16}$$

in place of Eq. (15), in line with the DA utility definition in Eq. (1), where the distribution of the returns is now conditional only on observed stock return data and not on the set θ.

2.3.1 Data

In order to study the problem of portfolio choice we utilize quarterly data from the U.S. market from January 1934 to September 2016 for the S&P 500 index (nominal index returns), the 3-month T-bill rate (which represents our risk-free asset) and the dividend price ratio (dividend yield). The latter is the predictor variable for the empirical part of this work. To calculate the annual dividend price ratio we sum up all the dividends paid throughout each year and divide them by the index level of the S&P500 at the end of the year. These datasets can be easily acquired by a number of sources as they are readily available online.[15]

2.3.2 i.i.d. returns

When investors ignore predictability in returns, they consider them to be i.i.d. and they use the following model to estimate next period's excess equity return:

$$x_t = (\mu - r) + \epsilon_t, \tag{17}$$

where x_t is the continuously compounded quarterly excess return of the S&P 500 index in quarter t and ϵ_t are i.i.d. disturbance terms distributed as $\epsilon_t \sim \mathcal{N}(0,\sigma)$.

[15] Our sources were the online platform of Bloomberg Professional Services (for the data on S&P 500 returns and the dividend price ratio), and the Federal Reserve (for the nominal interest rates).

The values for the parameters in Eq. (17) are $\mu = 0.02515$, $r = 0.00854$ and $\sigma = 0.08175$, also given in Table 1.

[Table 1 about here.]

Assuming investors are unaware of the true parameter value, we use an uninformative (diffuse) prior of the following type

$$p(\mu, \sigma) d\mu d\sigma \propto \frac{1}{\sigma} d\mu d\sigma, \tag{18}$$

while the joint posterior of the mean return μ and volatility σ is

$$p(\mu, \sigma|Y) \propto p(\mu, \sigma) \times L(\mu, \sigma|Y), \tag{19}$$

where L is the likelihood function. The following lemmas report the results for the case of i.i.d. returns (Lemma 1) and predictive returns (Lemma 2) where the VAR is used.

Lemma 1 *The distribution of the posterior moments for the case of i.i.d. returns is given by*

$$\sigma^2|Y \sim Inv - Gamma\left(\frac{N}{2}, \frac{1}{2}\sum_{i=1}^{N+1}(y_i - \overline{\mu})^2\right)$$
$$\mu|\sigma, Y \sim \mathcal{N}\left(\overline{\mu}, \frac{\sigma^2}{N}\right),$$

where Y is the observed asset return data, N is the sample size and $\overline{\mu}$ is the sample mean.

Proof. See Appendix D.1.

To construct the posterior predictive distribution for the i.i.d. returns of the risky asset we follow a standard sampling technique. We first sample once from the marginal posterior distribution $p(\sigma^2|Y)$ and then, we use the draw for σ to sample from the posterior distribution $p(\mu|\sigma, Y)$ which is now conditional on σ. We repeat this process to generate a sufficient number (i.e., 1,000,000) of pairs (μ, σ) to create return values and subsequently the posterior distribution for the returns of the risky asset, by sampling once for each pair (μ, σ). Details on the sampling procedure from the derived distributions for the mean and variance can be found in Appendix D.1.

2.3.3 Return predictability

In practice, asset returns do not follow a random walk. It is well documented that there are factors which can be used to predict part of the variability in asset returns (Lettau and Ludvigson, 2001; Campbell and Yogo, 2006; Ang and Bekaert, 2007; Cochrane, 2008). Investors use available information to predict future returns for optimal asset allocation problems. In this study, we replicate the prediction process using a VAR model where asset returns and the predictable variable are considered together. This results in time–varying investment opportunity sets which are conditional on the predictor variable in the VAR model. Investors' reaction entails the modification of their current investment proportions in the risky asset. We examined a number of financial variables[16] and chose the dividend yield (calculated as the dividend price ratio for the S&P 500 Composite Index) to be the driver of next period's equity return. The optimal lag-length was calculated to be one lag, as confirmed by both the Akaike and the Bayesian Information Criterion.

We then model the dividend-adjusted log excess returns of the risky asset as a first-order vector autoregression (VAR) of the following form

$$X_t = C + BX_{t-1} + E_t. \tag{20}$$

[16] To determine the variable that fits best with our data we test the following predictors: dividend yield (the sum of the dividends over a year divided by the level of the index at the end of the year); term spread (the difference between the 10-year T-bond and the 1-year T-bond); credit spread (the difference between Moodys BAA corporate bonds yield and its AAA equivalent); the 3-month T-bill; and the 10-year T-bond. The criteria for selecting the best fit are a) whether a variable enters the VAR as statistically significant and b) how much of the risky asset's excess return variability it explains.

In the model of Eq. (20) $X_t = \begin{pmatrix} y_t - r_{t-1} \\ (d/p)_{t-1} \end{pmatrix}$ where $y_t - r_{t-1} = x_t$ is the excess equity return, r_t is the risk–free rate and $(d/p)_{t-1}$ is the dividend price ratio, B is the (2 $\times$ 2) matrix of the autoregression coefficients, C is a (2×1) vector of the constant terms and E is a vector of *i.i.d.* normally distributed non-observable disturbance terms.

We use the lagged rate r_{t-1} to indicate that the value of the risk-free rate is known at the time of portfolio formation $t-1$, in contrast to the risky asset, whose return becomes known only at time t. When asset returns are not predictable, all elements of the matrix with the autoregressive coefficients B are not different from zero and returns are assumed to be i.i.d. As a result, the VAR model reduces to the i.i.d. return generator of Eq. (17). The vector autoregression in Eq. (20) is estimated using maximum likelihood estimation (MLE) and the results are reported in Table 2.

[Table 2 about here.]

Simulating return trajectories under the assumption that the dividend yield at time t can forecast asset returns at time $t+1$ we match the first two moments of the historical returns' distribution up to two to three significant figures. All coefficients of the matrix with the autoregressive parameters B are statistically significant at the 5% level, while both series (dividend yield and excess asset log returns) are stationary.

2.3.4 Parameter uncertainty with return predictability

The VAR in Eq. (20) can also be written in its compact form

$$X = BZ + E, \tag{21}$$

where $X = (X_1 \dots X_T)$ is a $(2 \times T)$ matrix with the number of observations T for the estimated variables, $Z = (z_0 \dots z_T)$ a $(3 \times T)$ matrix, $B \equiv (c\, B)$ is a (2×3) matrix of the auto-regressive coefficients and the constant terms, and the $E = (\epsilon_1 \dots \epsilon_T)$ is a $(2 \times T)$ matrix with the uncorrelated disturbance terms. A suitable uninformative prior is the Jeffreys prior given by

$$\begin{aligned} p(B, \Sigma) &= p(B)p(\Sigma) \\ &\propto |\Sigma|^{-(m+1)/2}, \end{aligned} \tag{22}$$

where $m = 2$ is the total number of regressors on the left-hand side of Eq. (21), $p(B)$ is constant and B is independent of Σ. We obtain the posterior density for the parameter matrix B and the covariance matrix of Eq. (21) by the following lemma.

Lemma 2 *The posterior distribution, $p(vec(B), \Sigma|X)$ for the coefficient matrix, B and the variance-covariance matrix, Σ conditional on data X is given by*

$$\Sigma|X \sim \mathcal{W}^{-1}((X - Z\hat{B})'((X - Z\hat{B}), T - n - 1)$$
$$vec(B)|\Sigma, X \sim \mathcal{N}(vec(\hat{B}), \Sigma^{-1}Z'Z),$$

where T is the number of observations in our sample and n is the number of predictor variables.

Proof. See Appendix (D.2).

Again, in order to sample from $p(vec(B), \Sigma|X)$, we sample first from $p(\Sigma|X)$ – the variance - covariance matrix – conditional on dataset X and then, given this draw, from the posterior distribution $p(vec(B), \Sigma|X)$ which will give a draw for the matrix of the VAR coefficients. The details of this process and the return generating procedure are presented in Appendix (D.2).

3 Asset allocation with DA preference

3.1 Buy-and-hold strategies

We first investigate the asset allocation problem for different investment horizons for buy-and-hold strategies. Here agents choose a static portfolio allocation strategy to the risky asset at the beginning of the investment period without optimal annual reallocation. This strategy results in the same allocation regardless of the investment horizon for an investor with power utility when returns follow an i.i.d. process (i.e., are normally distributed). The same result (i.e. fixed allocation to the risky asset) is achieved with an exponential utility function and lognormal returns or quadratic utility with normal returns. Our goal here is to explore the effects of introducing DA utility in conjunction with parameter uncertainty in place of power utility on the optimal asset allocation. We mainly focus on whether parameter uncertainty in a DA framework induces horizon effects (i.e. whether the long-term allocation to the risky asset is different to the short-term one).

[Figure 3 about here.]

Fig. 3 shows the optimal buy-and-hold portfolio allocations to the risky asset for a DA investor ($A = 0.44$ or $A = 0.30$) and an investor with power utility ($A = 1$; solid line) when returns are i.i.d. and parameter uncertainty is either considered (solid line) or ignored (dashed line). A DA investor who acknowledges parameter uncertainty will decrease the portfolio allocation to the risky asset with the investment horizon compared to one with power utility who will hold the same portfolio regardless of the horizon. This comes as a result of the variance's evolution of cumulative returns at difference horizons, which, under parameter uncertainty, grows faster than linearly, which is the case when parameter uncertainty is ignored. Under parameter uncertainty investors consider equity not as attractive in the long-run as in the case where parameter uncertainty is ignored which results in lower portfolio allocation to the risky asset. By Lemma 1 we see that the magnitude of the horizon effects depends on the available data incorporated in the model, in the following way: given σ, the variance of μ is inversely proportional to N (the sample size of risky asset return); subsequently, the larger the N, the lower the variance of μ and equivalently the smaller the uncertainty around its true value. Using a smaller sample would result in significantly lower allocation to the risky asset for an investor who considers uncertainty compared to one who ignores it, especially for longer horizons.

The incorporation of DA changes portfolio composition drastically. A DA investor ($A = 0.44$ or $A = 0.50$ will increase the investment proportion to the risky asset when they allocate wealth for longer periods. The effect of DA appears to be more powerful at short horizons ($T < 10$), as a DA investor who uses the DA utility function will hold significantly less equity compared to one with power utility. A DA investor who allocates optimally for horizons ranging between one and ten years will allocate between 20 and 50% of their portfolio to the risky asset (between 60 and 20% less equity compared to one with power utility), while an even more disappointment averse one ($A = 0.30$) will hold no more than 10 to 40 % equity for the same horizon. However, eventually investors with DA utility will allocate similarly to those with power utility as investment horizon increases. A DA investor appears to be very conservative in the short-run, while, when investing for long horizons, even a very DA one ($A = 0.30$, i.e. losses in her utility function are weighed more than three times more than gains) is willing to accept the additional risk by holding the risky asset in anticipation of higher terminal wealth, with lower volatility as a result of the longer investment horizon.

Predictability is critical in the case of a buy-and-hold investor, as one who takes into account predictability will allocate significantly larger weights to equity for longer investment horizons, as volatility does not grow proportionally to asset returns. The latter results in lower long-term volatility, compared to the short-term, thus making equities appear more attractive to an investor with a long-term outlook. In Fig. 4 we display optimal allocations to the risky asset for three levels of risk aversion (the ones most commonly used in relevant studies) and four levels of DA, among which is the value of $\frac{1}{\lambda}$ where λ is the loss aversion coefficient equal to $\lambda = 2.25$, as calculated by Tversky and Kahneman (1992). As expected, both risk aversion and DA affect the allocation to the risky asset as the more risk averse or disappointment averse an investor becomes, the lower this allocation will be. DA seems to be more relevant at shorter horizons (where the portfolio allocation to the risky asset for a DA investor is significantly lower compared to that for an investor with power utility - see left column of Fig. 4) and it is possible that this is due to short horizons' higher volatility rendering disappointment more likely to be experienced by an investor.

[Figure 4 about here.]

The cause behind the horizon effects we report for the buy-and-hold investor who uses the VAR to forecast equity returns can be traced in the evolution of return volatility. Long-term volatility is lower than in the case of i.i.d. returns, due to the correlation between the predictor variable and the predicted equity return.

[Figure 5 about here.]

More specifically, when we model returns as i.i.d., the two-period variance is equal to

$$var_{r_1,r_2} = var_{r_1} + var_{r_2} \Leftrightarrow \sigma_{r_1,r_2} = \sqrt{var_{r_1} + var_{r_2}}.$$

When returns are predictable, the covariance between equity returns and the predictor variable should be taken into consideration as well. The two-period variance is now equal to

$$var_{r_1,r_2} = var_{r_1} + var_{r_2} + 2cov(r_1, r_2).$$

Given that the covariance term in our VAR estimation is negative (see ρ and σ_{11}, σ_{22} in the first column of Table 2), the following holds:

$$var_{r_1} + var_{r_2} + 2cov(r_1, r_2) < var_{r_1} + var_{r_2}.$$

As a result, the long-term volatility for a buy-and-hold investor who uses the VAR is much smaller than that for the investor who uses the i.i.d. return generator, growing slower than linearly. In particular, under i.i.d. returns, the 40-year total volatility equals $0.1625\sqrt{40} = 1.02$ while the standard deviation for the 40-year total return as predicted by the VAR equals 0.5091, i.e., is half as much (see Fig. 5). This shows how the investment allocation in stocks can be affected (i.e., increase) by utilizing a variable that is believed to be able to predict stock returns.

The intuition behind this effect is twofold. On the one hand, assuming that dividend yield falls, its negative correlation with the expected stock return ($\rho < 0$; see Table 2) will drive the latter up. As dividend yield is now lower, actual stock returns will be lower as well given that b_{12} is positive. Higher expected returns (as predicted by the negative correlation between the dividend price ratio and risky asset returns) and lower realized returns (as indicated by the positive coefficient of risky asset returns) introduce a mean-reverting component, which, in turn, reduces the rate of increase of the variance, thus rendering equity more attractive at longer horizons. On the other hand, it is also possible that investors relying on a given strategy (in our case, the dividend yield) tend to develop illusion of control, if they grow overly confident in the ability of the strategy to generate precise predictions in terms of future returns; as this is bound to boost their overconfidence levels, it can lead them to assume higher risk in their investments by increasing their equity exposure (Odean, 1998; Gervais and Odean, 2001). This is expected to be further encouraged by the fact that investors whose outlook involves long horizons and/or buy-and-hold strategies end up monitoring their investments less frequently; the latter leads them to experience feelings of regret and/or disappointment equally less frequently, prompting them to view equity as less risky (since longer horizons entail fewer price fluctuations than short ones) and, thus tacitly encourage them to increase their exposure to it (Benartzi and Thaler, 1995).

When parameter uncertainty is incorporated (right column of Fig. 4), a DA investor who accounts for predictability will allocate a smaller proportion to the risky asset compared to one who ignores parameter uncertainty. When parameter uncertainty is incorporated, equities do not look as attractive as when parameter uncertainty is ignored, as a result of the higher volatility of equity returns; the latter is due to uncertainty dampening the correlation between the predictor variable and the dependent variable (i.e., equity return), which in turn increases the volatility faster than the case where parameter uncertainty is not considered. Expressing uncertainty about the parameters of the model is, in essence, equivalent to expressing uncertainty about the forecasting capacity of the predictor variable (i.e., the dividend price ratio). This uncertainty, in turn, can prompt investors to start viewing the VAR-process as potentially misspecified, thus rendering them more ambiguity-averse and leading them to reduce their exposure to equity investments (Maenhout, 2004). Under parameter uncertainty a DA investor will still hold larger weights for longer horizons compared to shorter ones but they will be significantly lower to those allocated when parameter uncertainty is ignored.

3.2 Dynamic strategies

We now present the results for the case of a DA investor who follows a dynamic strategy, reallocating her available wealth at the beginning of each period between the risk-free and the risky asset. An investor who allocates wealth dynamically considers the investment opportunity set for the whole remaining investment period $T - t$ and assigns the optimal weight to the risky asset knowing that she will have the chance to revise her strategy by the end of next period in case her expectations of the risky asset's return and volatility change. This is the difference between a dynamic and a myopic strategy in which investors follow a one-period forward-looking strategy.

3.2.1 Results with i.i.d. returns

With i.i.d. returns an investor who allocates dynamically ignoring parameter uncertainty, uses the normality assumption and the i.i.d. asset return generator with parameters equal to the historical annual mean and volatility of the S&P 500 ($\mu = 0.1045$, $\sigma = 0.1635$; see Table 1). As a result she has the same investment opportunity set at every horizon and the allocation to the risky asset does not change at different horizons (dashed line in Fig. 6).

[Figure 6 about here.]

3.2.2 Results with predictable returns

The left column of Fig. 7 reports optimal portfolio allocations for four different levels of the DA coefficient A and three levels of the risk-aversion coefficient γ at horizons $T-t$ between one and 40 years. The four levels of DA are the same as in the buy-and-hold case.

[Figure 7 about here.]

In these experiments, investors re-allocate their available wealth at the end of each year, taking into consideration the optimal solutions from the solved sub-problems at each horizon. For the same level of risk aversion, the more disappointment averse an investor is, the less she allocates to equities. The investment horizon effect of DA is visible by measuring the equity allocation at $T = 40$ and $t = 1$. The dynamic allocation to the risky asset drops as the investment horizon becomes shorter as a result of the lower per-period volatility for longer investment periods shown in Fig. 5. A moderately DA investor will still invest a significant part of her portfolio to equity even at very short horizons (dashed line in Fig. 7) while a more DA one will almost refrain from holding any units of the risky asset even for a relatively low level of risk aversion.

When investors believe returns to be forecastable, they use the VAR to predict next period's equity return and allocation drops with respect to the investment horizon for all four different values of A. As the investment horizon $T-t$ shortens, a DA investor who follows a dynamic strategy allocates a smaller proportion of her wealth to the risky asset, while a DA and risk-averse investor will hold no units of the risky asset as $T-t$ approaches zero. Again, investing dynamically to the risky asset for the short-run is not as attractive as for the long-run given the higher volatility per period of the former. As a consequence, the more disappointment averse an investor is the more likely it is to be affected by short-run volatility. This gives rise to horizon effects as investors try to hedge their portfolios at shorter horizons against a possible sharp move in the value of the independent variable (dividend yield).

3.2.3 Parameter uncertainty

Let us assume an investor who uses the i.i.d. return generator and considers uncertainty in the model's parameters. In this case she will exhibit slightly different portfolio allocations compared to when the model's parameters are treated as known. Fig. 6 shows that both a DA investor and one who uses the power utility function will increase their portfolio allocation to the risky asset with the investment horizon (solid line) to eventually hold a portfolio position very similar to one who ignores parameter uncertainty (dashed line). Investing for a longer horizon appears to be more risky than holding the risky asset in the short-run as a result of the lower per period volatility of the latter. As a result, an investor who invests dynamically with a shorter term outlook will hold slightly more equity in their portfolio compared to one who invests for a longer horizon.

Turning to the case of predictability, the right column of Fig. 7 reports results which reflect optimal allocations to the risky asset for investors who rebalance their portfolios by predicting asset returns based on the dividend yield when parameter uncertainty is accounted for. These plots reveal mainly two facts; first, equity allocation is, in general, lower compared to the case of an investor who ignores parameter uncertainty and second, the impact of DA appears again to be more powerful at shorter horizons (up to 10 years) while for longer ones allocation lines become relatively flat. When we express uncertainty about the parameters of the VAR, we use the posterior predictive distribution in Lemma 2 in place of the VAR model with fixed parameters as stated in Eq. (20). In this case, instead of simulating future return paths conditioning on fixed values for the model parameters (constant terms, matrix of AR coefficients and variance-covariance matrix), we sample from their posterior distributions obtaining each time a new set of parameters which is conditional only on observed data.

The results exhibit a similar pattern to the ones in the left column of Fig. 7. The more disappointment averse and risk averse an investor grows, the lower the equity allocation will be at different investment horizons. As in the case of a DA investor who follows a buy-and-hold strategy, the choice of the DA level affects mainly the dynamic allocation at longer horizons, while, as $T-t$ approaches zero, the allocation lines exhibit converging behaviour. The underlying cause for this behaviour can be found in the way the mean return and variance change over time. Investors' uncertainty about the predictive capacity of the dividend yield makes equity returns generated by the VAR more similar to those derived by the i.i.d. return generator, as the negative correlation between the equity return and the predictor variable is weaker. The latter results in higher long-term per-period volatility, which explains the lower allocation to the risky asset compared to the left column of Fig. 7 where parameter uncertainty is ignored.

In other words, there are cases where parameter uncertainty makes investors sceptical about whether investment opportunities actually change over time. Subsequently, they doubt that higher or lower equity allocations will result in more optimal portfolios. In this case, the allocation will be similar at different horizons compared with the case where parameter uncertainty is ignored.

4 Concluding remarks

Disappointment aversion (DA) is a critical factor in portfolio choice, as its variation can affect portfolio compositions drastically. Our experiments suggest that in the context of an utility maximization problem, a DA investor would allocate lower weights to equities compared to an investor who uses a standard CRRA power utility function.

For a buy-and-hold investor, we detect horizon effects for a DA investor who uses either of the return generators and accounts for or ignores parameter uncertainty. For an investor who uses the VAR to predict equity returns, the examination of the evolution of risky asset's return volatility throughout the investment horizon reveals that it grows slower than linearly - the case when the i.i.d. return generator is used. This seems to offer a plausible explanation for the observed horizon effects while in the case of a VAR with parameter uncertainty, the additional uncertainty is expressed as additional volatility in risky assets return which in turn decreases the allocation to the risky asset.

Focusing on dynamic investing, we examined cases where investors believe returns are i.i.d. or forecastable (through the dividend yield), and parameter uncertainty is ignored as well as incorporated in the asset allocation framework. Overall, the decision of whether one will invest under predictable or non-predictable returns is critical in the presence of DA, as the equity allocation changes measurably between the two alternatives. When predictability is taken into consideration, the distribution of the future returns generated by the VAR is significantly different from that of i.i.d. returns, due to the presence of correlation between the dividend price ratio and the return of the risky asset. With i.i.d. returns, horizon effects arise when parameter uncertainty is taken into consideration. Investors allocate smaller proportions to stocks for shorter horizons after accounting for the increased variance in equity returns.

Finally, the incorporation of parameter uncertainty in the DA framework with predictability changes equity allocations drastically. Overall, it is beneficial to be examined as a special case in a portfolio model, as frameworks which do not account for this may generate portfolios with too large equity allocations. When model parameters are taken as uncertain, a DA investor will still allocate larger weights to stocks at longer horizons. Nevertheless, the difference between a long-term and a short-term equity weight is smaller compared to the case where parameter uncertainty is ignored, as a result of the doubts investors cast on the predictive power of the dividend yield.

Our results overall suggest that the prevalence of DA among investors tends to prompt them to reduce their exposure to equity, with the degree of said exposure varying conditional on their view of the return-generation process and the uncertainty innate in the latter's parameters. With retail investors being more likely to be subject to DA given their overall susceptibility to behavioural biases – see Barber et al. (2009) –, financial literacy initiatives could raise awareness of the latter's effects in order to help individual investors improve on their trading decisions. Similarly, finance practitioners with regular client-interface (such as brokers and financial advisors) could consider controlling for DA when assessing their clients' risk profile in order to gain better understanding of their clients' disposition towards trading and help them improve on their investments. What is more, our results further denote that DA can help explain the relative reluctance of investors to hold equities (thus contributing to the debate on the equity premium puzzle) and re-enter the market if they have exited it previously at a loss.

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APPENDICES

A Static Portfolio Allocation

In this appendix we revisit the static asset allocation problem, with the purpose of updating Ang et al. (2005) using the most recent data sample (1934–2016). We start by formulating the first order condition (FOC, thereafter) for the single-period investing, which is given by:

Proposition 3 *The FOC for the maximization of $U(\mu)$ in Eq. (1) is given by*

$$\mathbb{E}\left[\frac{dU(\mu_W)}{d\mu_W} X\, \mathbf{1}_{[W \leq \mu_W]}\right] + A\mathbb{E}\left[\frac{dU(\mu_W)}{d\mu_W} X\, \mathbf{1}_{[W > \mu_W]}\right] = 0, \;\; \alpha \neq 0, \tag{23}$$

where $X = e^y - e^{r_f}$ is the return of equity over that of the riskless asset.

Proof. We calculate the FOC for the single-period case. Let

$$W = \alpha(e^y - e^r) + e^r = \alpha X^e + e^r, \tag{24}$$

where α is the portfolio weight allocated to the risky asset, y is the risky asset's return, r is the risk-free interest rate known at the time of every investment decision and X^e is the equity premium. Considering an arbitrary utility function U, we extend it to define μ (i.e., the certainty equivalent of wealth) in the following way:

$$U(\mu) = \frac{1}{K}\Big\{\mathbb{E}(U(W))\mathbf{1}_{W\leq\mu} + A\mathbb{E}(U(W))\mathbf{1}_{W>\mu}\Big\}, \tag{25}$$

where $K = \mathbb{E}(\mathbf{1}_{W\leq\mu}) + A\mathbb{E}(\mathbf{1}_{W>\mu})$. Now, maximizing over α, the Eq. (26) derives:

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{1}{K}\left\{\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W\leq\mu} + A\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W>\mu})\right\} - \frac{U(\mu)}{K}\left\{\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W\leq\mu}) + \frac{d}{d\alpha}A\mathbb{E}(U(W)\mathbf{1}_{W>\mu})\right\}, \tag{26}$$

where the second part of the last equation follows after plugging Eq. (25) into Eq. (26) and simplifying the expression. We notice that

$$\mu = \alpha X + R \Leftrightarrow X = \frac{\mu - R}{\alpha}, \tag{27}$$

therefore the derivatives of the expected values can be expressed as follows,

$$\frac{d}{d\alpha}\mathbb{E}(U(W))\mathbf{1}_{W>\mu} = \frac{d}{d\alpha}\left(\int_{\frac{\mu-R}{\alpha}}^{\xi} f(X)U(\alpha X + R)dX + \int_{\xi}^{\infty} f(X)U(\alpha X + R)dX\right), \tag{28}$$

which by the Leibniz rule can be written as

$$\begin{aligned}&\int_{\frac{\mu-R}{\alpha}}^{\infty} f(X)\frac{dU(\alpha X + R)}{d\alpha}XdX - f\left(\frac{\mu - R}{\alpha}\right)U(\mu)\frac{d}{d\alpha}\left(\frac{\mu - R}{\alpha}\right) = \\ &\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W>\mu}\right) - f\left(\frac{\mu - R}{\alpha}\right)U(\mu)\frac{d}{d\alpha}\left(\frac{\mu - R}{\alpha}\right),\end{aligned} \tag{29}$$

where $f(X)$ is the normal probability density function. Similarly, the Eqs. (30), (31) and (32) are calculated,

$$\frac{d}{d\alpha}\mathbb{E}(U(W)\mathbf{1}_{W\leq\mu}) = \mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W\leq\mu}\right) + f\left(\frac{\mu - R}{\alpha}\right)U(\mu)\frac{d}{d\alpha}\left(\frac{\mu - R}{\alpha}\right), \tag{30}$$

$$\frac{d}{d\alpha}\mathbb{E}(\mathbf{1}_{W\leq\mu}) = \frac{d}{d\alpha}\int_{-\infty}^{\frac{\mu-R}{\alpha}} f(X)dX = \frac{d}{d\alpha}\left(\frac{\mu - R}{\alpha}\right)f\left(\frac{\mu - R}{\alpha}\right) \tag{31}$$

and

$$\frac{d}{d\alpha}\mathbb{E}(\mathbf{1}_{W>\mu}) = \frac{d}{d\alpha}\int_{\frac{\mu-R}{\alpha}}^{\xi} f(X)dX = -\frac{d}{d\alpha}\left(\frac{\mu - R}{\alpha}\right)f\left(\frac{\mu - R}{\alpha}\right). \tag{32}$$

Substituting Eqs. (30) to (31) into (29), we take:

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{1}{K}\left[\mathbb{E}\left(\frac{X dU(W)}{dW}\mathbf{1}_{W\le\mu}\right) + U(\mu) f\left(\frac{\mu-R}{\alpha}\right)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right) + A\mathbb{E}\left(\frac{X dU(W)}{dW}\mathbf{1}_{W>\mu}\right)\right.$$
$$\left. - AU(\mu) f\left(\frac{\mu-R}{\alpha}\right)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)\right] - \frac{U(\mu)}{K}\left[\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right) f\left(\frac{\mu-R}{\alpha}\right)\right.$$
$$\left. - Af\left(\frac{\mu-R}{\alpha}\right)\frac{d}{d\alpha}\left(\frac{\mu-R}{\alpha}\right)\right] = \frac{1}{K}\left[\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W\le\mu}\right) + A\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W>\mu}\right)\right]. \tag{33}$$

The FOC is now given by

$$\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W\le\mu}\right) + A\mathbb{E}\left(X\frac{dU(W)}{dW}\mathbf{1}_{W>\mu}\right) = 0, \quad \alpha \neq 0. \tag{34}$$

■

Eqs. (1) and (23) have to be solved simultaneously, since $\mu_W = \mu_W(A,\alpha)$ is itself a function of α and A. A way to proceed with the solution of the resulting system of equations is to discretize it so as to convert it into a discrete one. For more details on this process, see Appendix B.

B Solution to DA Portfolio Choice Problem

This appendix describes the discretization procedure of the DA asset allocation problem and the solution to the system of simultaneous equations (including the DA utility function and its optimization expression).

Analytically, to solve the system of Eqs. (10) and (13), the Gaussian Quadrature method is used. Since a lognormal distribution for the returns is assumed, the logarithmic returns are normally distributed. Under this assumption the numerical scheme of Gauss–Hermite is used to convert integrals of exponential expressions into the form of

$$\int_{-\infty}^{\infty} exp(-x^2)f(x)dx \approx \sum_{i=1}^{N} f(x_i)w_i, \tag{35}$$

where $\{x_i\}_{i=1}^N$ are the discrete points the integral in Eq. (35) is calculated at and $\{w_i\}_{i=1}^N$ are the corresponding weights. The points x_i, known as abscissae, are the roots of the Hermite polynomials, while the weights w_i are derived after a relevant transformation. Based on the discretization procedure, we write Eq. (10) as

$$\mu_t = W_t \left(\prod_{i=t+1}^{T-1} \mu_i^* \right) \left[\frac{\sum_{s=1}^{M} p_s W_{s,t+1}^{1-\gamma} + A \sum_{s=M+1}^{N} p_s W_{s,t_1+1}^{1-\gamma}}{P(W_{t+1} \leq \mu_t) + AP(W_{t+1} > \mu_t)} \right]^{1/(1-\gamma)}, \tag{36}$$

and Eq. (34), the FOCs of the problem, as

$$\sum_{s=1}^{M} p_s W_{s,t+1}^{-\gamma} X_{s,t+1} + A \sum_{s=M+1}^{N} p_s W_{s,t+1}^{-\gamma} X_{s,t+1} = 0, \tag{37}$$

where $X_{t+1} = e^{y,t+1} - e^{r,t}$ is the excess return of the risky asset over the period $t, t+1$, and s takes one of the values 1 to N, where N is the number of risky asset's discrete states. We notice that the outcomes are split to the two sums with respect to their relationship to the certainty equivalent. Given that the discrete return states are ordered from smallest to largest, the first sum takes on all the discrete outcomes that lie below μ_t while the second one takes those above the certainty equivalent and scales them down via the DA coefficient A.

The solution to the Eqs. (36) and (37) yields the portfolio weight α that maximizes the DA utility and the value of the certainty equivalent for each corresponding period. This solution is non-trivial, and thus we need to follow an algorithmic procedure similar to that in Ang et al. (2005). Considering the N states for the excess return at time $t+1$, $y_{t+1} - r_t$, we construct $N-1$ ordered intervals for the portfolio return as follows:

$$\begin{aligned}&\left[(\alpha X_1 + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}, (\alpha X_2 + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}\right), \ldots, \\ &\left[(\alpha X_{N-1} + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}, (\alpha X_N + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}\right).\end{aligned} \tag{38}$$

Assuming that the certainty equivalent μ_W lies in the interval defined by the return states i and $i+1$, i.e., $[(\alpha X_i + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j}, (\alpha X_{i+1} + e^r)\prod_{j=t+1}^{T-1}\mu^*_{W_j})$ where $1 < i \leq N$, $\prod_{j=t+1}^{T-1}\mu^*_{W_j}$ is the indirect utility of wealth and α satisfies the FOC, (37),

$$\sum_{s:W_s\prod_{j=t+1}^{T-1}\mu^*_{W_j}\leq\alpha^* X_i + e^r} p_s W_s^{-\gamma} X_s + A \sum_{s:W_s\prod_{j=t+1}^{T-1}\mu^*_{W_j}>\alpha^* X_{i+1} + e^r} p_s W_s^{-\gamma} X_s = 0, \tag{39}$$

where α^* is now the optimal value for the portfolio weight. As Eq. (36) indicates, the probabilities for the outcomes above the certainty equivalent should be downweighted. Therefore, the corresponding probabilities are multiplied by the DA coefficient A and then divided by the sum of all probabilities related to the possible return states so as to add up to one. The new probability distribution is defined as

$$\pi_s = \frac{p_s 1_{\{s\leq i\}} + A p_s 1_{\{s\geq i+1\}}}{\sum_{s=1}^{i} p_s + A\sum_{s=i+1}^{N} p_s},\ 1 < s \leq N. \tag{40}$$

Given initial guess i for the state of the certainty equivalent we solve for α^* and

μ_W^* which is now stated as

$$\mu_{W_i} = \left(\pi_s \sum_{s=1}^{N} (W_s)^{1-\gamma} \prod_{j=t+1}^{T-1} \mu_{W_j}^* \right)^{\frac{1}{1-\gamma}}, \tag{41}$$

where the second term of Eq. (36) is absorbed by the changed probability distribution in Eq. (40). In case $\mu_{W_i}^*$ lies within the interval defined by our initial guess

$$\mu_{W_i} \in \left[(\alpha_i X_i + e^r) \prod_{j=t+1}^{T-1} \mu_{W_j}^*, (\alpha_i X_{i+1} + e^r) \prod_{j=t+1}^{T-1} \mu_{W_j}^* \right), \tag{42}$$

$\mu_W^* = \mu_{W_i}$, $\alpha^* = \alpha_i$ and i is the optimal state for the problem. If the condition in Eq. (42) is not satisfied we perform a binary search given ordered return intervals, until α falls within the right interval.

C Different Proofs

This appendix contains the proofs of propositions and the theorem stated in the main body of this paper.

C.1 Proof of Proposition 1

We define the DA utility function for the dynamic asset allocation problem in accordance with Ang et al. (2005) as follows:

At time $t = T-1$ we have

$$U(\mu_{T-1}) = \frac{1}{K_{T-1}}\Big[\mathbb{E}_{T-1}(U(W_{T-1}R_T(\alpha_{T-1}))\mathbf{1}_{\{W_{T-1}R_T(\alpha_{T-1})\leq\mu_{T-1}\}}) + A\mathbb{E}_{T-1}(U(W_{T-1}R_T(\alpha_{T-1}))\mathbf{1}_{\{W_{T-1}R_T(\alpha_{T-1})>\mu_{T-1}\}})\Big]. \quad (43)$$

Continuing recursively, at time $t = T-2$ the DA utility is defined as

$$U(\mu_{T-2}) = \frac{1}{K_{T-2}}\Big[\mathbb{E}_{T-2}(U(W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1}))\mathbf{1}_{\{W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1})\leq\mu_{T-2}\}}) + A\mathbb{E}_{T-2}(U(W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1}))\mathbf{1}_{\{W_{T-2}R_{T-1}(\alpha_{T-2})R_T(\alpha^*_{T-1})\leq\mu_{T-2}\}})\Big]. \quad (44)$$

Eventually, at time t, we will have

$$U(\mu_t) = \frac{1}{K_t}\Big[\mathbb{E}_t(U(W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}))\mathbf{1}_{\{W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}\leq\mu_t\}}) + A\mathbb{E}_t(U(W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}))\mathbf{1}_{\{W_tR_{t+1}(\alpha_t)Q^*_{t+1,T}>\mu_t\}})\Big], \quad (45)$$

where $Q^*_{t+1,T} = R_{t+2}(\alpha^*_{t+1})\cdots R_T(\alpha^*_{T-1})$ is the optimal aggregate return between $t+1$ and T that maximizes the correspodning utility of wealth $U(W)$.

We next calculate the optimization condition for the multiperiod dynamic problem. In this case, we have T periods and our investor maximizes the utility of wealth

$$\max_{\alpha_0,\alpha_1,\cdots,\alpha_{T-1}} \mathbb{E}_0[U(W_T)], \quad (46)$$

where the wealth is given by $W_t = R_t(\alpha_{t-1})W_{t-1}$, where $R_t(\alpha_{t-1}) = \alpha_{t-1}(e^{y_t} - e^{r_{t-1}}) + e^{r_{t-1}}$. Considering we are at time $T-1$, we follow Eq. (33) and by adding time subscripts we end up with the following expression:

$$\begin{aligned}\frac{dU(\mu_{T-1})}{d\mu_{T-1}}\frac{d\mu_{T-1}}{d\alpha_{T-1}} &= \frac{1}{K_{T-1}}\Bigg[\mathbb{E}_{T-1}\bigg(X_T\frac{dU(W_T)}{dW}\mathbf{1}_{W_T\leq\mu_{T-1}}\bigg)\\ &\quad + A\mathbb{E}_{T-1}\bigg(X_T\frac{dU(W_T)}{dW}\mathbf{1}_{W_T>\mu_{T-1}}\bigg)\Bigg]\\ &= \frac{1}{K_{T-1}}\Bigg[\mathbb{E}_{T-1}\bigg(W_{T-1}\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T\leq\mu_{T-1}}\bigg)\\ &\quad + A\mathbb{E}_{T-1}\bigg(W_{T-1}\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T>\mu_{T-1}}\bigg)\Bigg].\end{aligned} \tag{47}$$

But at time $T-1$, the terms in W_{T-1} become known and eventually W_{T-1} can be taken outside the expectation term leading to the following FOC

$$\mathbb{E}_{T-1}\bigg(\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T\leq\mu_{T-1}}\bigg)+A\mathbb{E}_{T-1}\bigg(\frac{dU(R_T(\alpha_{T-1}))}{dW}X_T\mathbf{1}_{W_T>\mu_{T-1}}\bigg) = 0. \tag{48}$$

Moving backwards to time t, the wealth W_T can be expressed as

$$W_T = Q^*_{t+1,T}R_{t+1}(\alpha_t)W_t \tag{49}$$

and Eq. (47) can be rewritten in the following way

$$\begin{aligned}\frac{dU(\mu_t)}{d\mu_t}\frac{d\mu_t}{d\alpha_t} = \frac{1}{K_t}\Bigg[&\mathbb{E}_t\bigg(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\leq\mu_t}\bigg)\\ &+ A\mathbb{E}_t\bigg(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\bigg)\Bigg],\end{aligned} \tag{50}$$

and the FOC is as follows

$$\begin{aligned}&\mathbb{E}_t\bigg(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T\leq\mu_t}\bigg)\\ &\qquad + A\mathbb{E}_t\bigg(\frac{dU(W_T)}{dW}Q^*_{t+1,T}R_{t+1}(\alpha_t)W_tX_{t+1}\mathbf{1}_{W_T>\mu_t}\bigg) = 0.\end{aligned} \tag{51}$$

C.2 Proof of Proposition 2

We formulate the FOCs for the optimization problem by performing the substitution of the value $R_{i+1}(\alpha_i^*), i = t, t+1\cdots, T-1$ with the certainty equivalent for the same period, μ_i^*. Using this approach, we keep the dimension of the state space constant with time, which allows us to solve the problem computationally in reasonable time. Recalling the implicit definition for the certainty equivalent and the expression for $Q^*_{t+1,T}$ we have that

$$U(\mu_t) = \frac{1}{K_t}\left[\mathbb{E}_t(U(W_t)\mathbf{1}_{W_T\leq\mu_t}) + A\mathbb{E}_t(U(W_t)\mathbf{1}_{W_T>\mu_t})\right]$$
$$= \frac{1}{K_t}\left[\mathbb{E}_t(U(Q^*_{t+1,T}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{W_T\leq\mu_t}) + A\mathbb{E}_t(U(Q^*_{t+1,T}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{W_T\leq\mu_t})\right]$$
$$= \frac{1}{K_t}\left[\mathbb{E}_t\left(U(\mu^*_{T-1}\dots\mu^*_{t+1}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}\}}\right)\right.$$
$$\left.+ A\mathbb{E}_t\left(U(\mu^*_{T-1}\dots\mu^*_{t+1}R_{t+1}(\alpha_t)W_t)\mathbf{1}_{\{R_{t+1}(\alpha_t)>\frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}\}}\right)\right]. \quad (52)$$

In Eq. (52), the product $\mu^*_{T-1}\dots\mu^*_{t+1}W_t$ is known at time t and thus, for simplicity, it can be written outside the expectation terms as a function of the utility of wealth $U(\cdot)$. This transformation is as follows:

$$U(\mu_t) = \frac{f(U(\prod_{i=t+1}^{T-1}\mu^*_iW_t))}{K_t}\left[\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{\{R_{t+1}(\alpha_t)\leq\xi_t\}})\right.$$
$$\left.+ A\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{\{R_{t+1}(\alpha_t)>\xi_t\}})\right], \quad (53)$$

where $\xi_t = \frac{\mu_t}{\mu^*_{T-1}\cdots\mu^*_{t+1}W_t}$ and $K_t = \mathbb{E}_t(\mathbf{1}_{W_T\leq\mu_t}) + A\mathbb{E}_t(\mathbf{1}_{W_T>\mu_t})$. Hence, we obtain

$$\frac{1}{f(U(\mu^*_{T-1}\dots\mu^*_{t+1}W_t))}\frac{dU(\mu_t)}{d\mu_t}\frac{d\mu_t}{d\alpha_t} = -\frac{1}{K_t^2}\frac{dK_t}{d\alpha_t}K_tU(\mu_t)+\frac{1}{K_t}\left[\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)\leq\xi_t}\right.$$
$$\left.+ A\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right]$$
$$= \frac{1}{K_t}\left[A\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t} + \frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)\leq\xi_t}\right]$$
$$- \frac{U(\mu_t)}{K_t}\left[A\frac{d}{d\alpha_t}\mathbb{E}_t(\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}) + \frac{d}{d\alpha_t}\mathbb{E}_t(\mathbf{1}_{R_{t+1}(\alpha_t)\leq\xi_t})\right]. \quad (54)$$

But we know that

$$R_{t+1}(\alpha_t) > \xi_t \Leftrightarrow \alpha_tX_{t+1} + e^{r_t} > \xi_t \Leftrightarrow X_{t+1} > \frac{\xi_t - e^{r_t}}{\alpha_t} \equiv \xi,\ \ \alpha > 0, \quad (55)$$

therefore, we can express the derivatives as in the proof of Proposition 1. More specifically, the first term of Eq. (54) is

$$
\begin{aligned}
\frac{d}{d\alpha_t}\mathbb{E}_t(U(R_{t+1}(\alpha_t))\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t} &= \frac{d}{d\alpha_t}\int_{\xi}^{\infty} U(R_{t+1}(\alpha_t))F(X_{t+1})dX_{t+1} \\
&= \frac{d}{d\alpha_t}\int_{\xi}^{\infty} U(\alpha_t X_{t+1} + e^{r_t})F(X_{t+1})dX_{t+1} \\
= \frac{d}{d\alpha_t}\int_{\xi}^{\xi'} U(\alpha_t X_{t+1}+e^{r_t})F(X_{t+1})dX_{t+1} &+ \frac{d}{d\alpha_t}\int_{\xi'}^{\infty} U(\alpha_t X_{t+1}+e^{r_t})F(X_{t+1})dX_{t+1} \\
= \int_{\xi}^{\xi'} \frac{d}{d\alpha_t}U(\alpha_t X_{t+1} + e^{r_t})X_{t+1}F(X_{t+1})dX_{t+1} &- F(\xi)U\left(\alpha_t\frac{\xi_t - e^{rt}}{\alpha_t} + e^{r_t}\right)\frac{d\xi}{d\alpha_t} \\
&+ \int_{\xi'}^{\infty} \frac{d}{d\alpha_t}U(\alpha_t X_{t+1} + e^{r_t})X_{t+1}F(X_{t+1})dX_{t+1} \\
&= \int_{\xi}^{\infty} \frac{d}{d\alpha_t}U(\alpha_t X_{t+1} + e^{r_t})X_{t+1}F(X_{t+1})dX_{t+1} - F(\xi)U(\xi_1)\frac{d\xi}{d\alpha_t} \\
&= \mathbb{E}_t\left(X_{t+1}\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right) - F(\xi)U(\xi_t)\frac{d\xi}{d\alpha_t}.
\end{aligned}
\tag{56}
$$

Expressing the remaining terms of Eq. (54) in the same way we obtain the following result:

$$
\begin{aligned}
&\frac{1}{f(U(\mu^*_{T-1}\cdots\mu^*_{t+1}W_t))}\frac{dU(\mu_t)}{d\mu_t}\frac{d\mu_t}{d\alpha_t} \\
&= \frac{1}{K_t}\left[\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)\leq\xi_t} + A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right)\right)\right],
\end{aligned}
$$

which considering the FOC of the expression above yields

$$
\begin{aligned}
&\frac{dU(\mu_t)}{d\alpha_t} = 0 \Leftrightarrow \\
&\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)\leq\xi_t}\right) + A\mathbb{E}_t\left(\frac{dU(R_{t+1}(\alpha_t))}{d\alpha_t}X_{t+1}\mathbf{1}_{R_{t+1}(\alpha_t)>\xi_t}\right) = 0,
\end{aligned}
\tag{57}
$$

with $t = T-1, \ldots, 0$. Comparing Eqs. (57) and (51), one can see the advantage of using the former over the latter as it involves only one uncertain variable, the return of the portfolio between t and $t+1$.

C.3 Proof of Theorem 1

We prove that for the critical level of disappointment aversion, A^*, any A below this value induces non-participation ($\alpha < 0$) while any A larger than A^* leads to positive portfolio allocation. Let $\mu = \mu_W(A, \alpha)$, with

- $\mu(A, :) \in \mathbf{C}^1, \forall A \in [0, 1]$,
- $\frac{d\mu(A,0)}{d\alpha} = \xi(A) \leq 0, \forall A \in [0, 1]$,
- $\mathbb{E}(X) > 0$ and $\mathbb{E}(X\mathbf{1}_{W \geq \xi(A)}) > 0$, where $X = e^y - e^r$ is the excess return of the equity over the bond.

Then, setting

$$A^* = \frac{\mathbb{E}(X\mathbf{1}_{W \geq \xi(A)})}{\mathbb{E}(X\mathbf{1}_{W < \xi(A)})}, \tag{58}$$

we have the following:

- For every $A \leq A^*$, $\quad \alpha^* = 0$;
- For every $A > A^*$, $\quad \alpha^* > 0$,

where α^* is the portfolio weight which maximizes $\mu(A, \alpha)$ for a given level of A.

Proof. We have that

$$W = \alpha(e^y - e^r) + e^r \stackrel{\Delta}{=} \alpha X + R, \tag{59}$$

which as $\alpha \to 0$ tends to R. The expected value of Eq. (59) equals $\mathbb{E}(W) = \alpha\mathbb{E}(X) + r$. From the definition of the DA utility we have

$$\lim_{\alpha \to 0} U(\mu) = \lim_{\alpha \to 0} \frac{\mathbb{E}(U(W))\mathbf{1}_{W \leq \mu} + A\mathbb{E}(U(W))\mathbf{1}_{W > \mu}}{\mathbb{P}(W \leq \mu) + A\mathbb{P}(W > \mu)}, \tag{60}$$

which given that both the utility function and the certainty equivalent are $\mathbb{C}^1$-functions can be written as

$$U(\mu(A, 0)) = \frac{\mathbb{E}(U(r)\mathbf{1}_{r \leq \mu(A,:)} + A\mathbb{E}(U(R))\mathbf{1}_{r > \mu(A,:)}}{\mathbb{P}(r \leq \mu(A, :)) + A\mathbb{P}(r > \mu(A, :))} \implies \mu(A, 0) = r. \tag{61}$$

The last equality follows from the fact that the certainty equivalent U is a $1-1$ function. We now examine the behaviour of the function μ around zero. Thus, we consider two cases, one where α approaches zero from negative values and one where it approaches zero from positive ones.

- $a < 0$; from the FOCs for the optimization problem we have

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{1}{K}\left\{\mathbb{E}\left(X\frac{dU}{dW}\mathbf{1}_{X\geq(\mu-r)/\alpha}\right) + A\mathbb{E}\left(X\frac{dU}{dW}\mathbf{1}_{X<(\mu-r)/\alpha}\right)\right\}, \quad (62)$$

where $K = \mathbb{P}(W \leq \mu) + A\mathbb{P}(W > \mu) = \mathbb{P}(X \geq \frac{\mu-r}{\alpha}) + A\mathbb{P}(X < \frac{\mu-r}{\alpha})$. Therefore,

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{1}{K}\left\{\int_{(\mu-r)/\alpha}^{+\infty} X\frac{dU(\alpha X+r)}{dW}f(X)dX + A\int_{-\infty}^{(\mu-r)/\alpha} X\frac{dU(\alpha X+r)}{dW}f(X)dX\right\}. \quad (63)$$

We have that

$$\frac{d\mu(A,0)}{d\alpha} = \xi(A) \equiv \xi \quad (64)$$

with

$$\lim_{\alpha\to 0}\frac{\mu(A,\alpha)-\mu(A,0)}{\alpha} = \xi \Leftrightarrow \lim_{\alpha\to 0}\frac{\mu(A,\alpha)-r}{\alpha} = \xi. \quad (65)$$

We now define

$$B(A,\alpha) = \begin{cases} \frac{\mu(A,\alpha)-r}{\alpha} & \text{if } \alpha <> 0 \\ \xi & \text{if } \alpha = 0 \end{cases} \quad (66)$$

where B is a continuous function. Therefore, Eq. (62) can be rewritten as

$$\frac{dU}{d\mu}\frac{d\mu}{d\alpha} = \frac{\int_{B(A,\alpha)}^{+\infty} X\frac{dU(\alpha X+r)}{dW}f(X)dX + A\int_{-\infty}^{B(A,\alpha)} X\frac{dU(\alpha X+r)}{dW}f(X)dX}{\int_{B(A,\alpha)}^{+\infty} f(X)dX + A\int_{-\infty}^{B(A,\alpha)} f(X)dX}, \alpha \leq 0. \quad (67)$$

Subsequently we have,

$$\begin{aligned} \frac{du(\mu(A,0))}{d\mu}\lim_{\alpha\to 0^-}\frac{d\mu(A,\alpha)}{d\alpha} &= \frac{\int_{\xi}^{+\infty} X\frac{dU(r)}{dW}f(X)dX + A\int_{-\infty}^{\xi} X\frac{dU(r)}{dW}f(X)dX}{\int_{\xi}^{+\infty} f(X)dX + A\int_{-\infty}^{\xi} f(X)dX} \Leftrightarrow \\ \lim_{\alpha\to 0^-}\frac{d\mu(A,\alpha)}{d\alpha} &= \frac{\mathbb{E}(X\mathbf{1}_{W\geq\xi}) + A\mathbb{E}(X\mathbf{1}_{W<\xi})}{\mathbb{P}(X\geq\xi) + A\mathbb{P}(X<\xi)}. \end{aligned} \quad (68)$$

Since $\xi(A) \leq 0$ and $A \in [0,1]$ we have that

$$X\mathbf{1}_{W\geq\xi} + AX1_{W<\xi} \geq X \Rightarrow \lim_{\alpha\to 0^-}\frac{d\mu(A,\alpha)}{d\alpha} > 0. \quad (69)$$

Thus, there is $\epsilon > 0$ such that $\frac{d\mu(A,\alpha)}{d\alpha} > 0$ for every $\alpha \in (-\epsilon, 0) \Rightarrow \mu(A,\alpha)$, is strictly increasing with respect to α in $(-\epsilon, 0)$.

- $\alpha > 0$; in the case where zero is approached from the right we have

$$\frac{du(\mu(A,0))}{d\mu} \lim_{\alpha \to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} = \frac{\int_{-\infty}^{\xi} X \frac{dU(r)}{dW} f(X) dX + A \int_{\xi}^{+\infty} X \frac{dU(r)}{dW} f(X) dX}{\int_{-\infty}^{\xi} f(X) dX + A \int_{\xi}^{+\infty} f(X) dX} \Leftrightarrow$$
$$\lim_{\alpha \to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} = \frac{\mathbb{E}(X \mathbf{1}_{W \le \xi}) + A \mathbb{E}(X \mathbf{1}_{W > \xi})}{\mathbb{P}(X \le \xi) + A \mathbb{P}(X > \xi)}, \tag{70}$$

which leads to

$$\lim_{\alpha \to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} = \frac{\mathbb{E}(X \mathbf{1}_{W \le \xi}) + A^* \mathbb{E}(X \mathbf{1}_{W > \xi})}{K}, \tag{71}$$

since $A < A^*$ and the expected value of the return premium, X is positive. Now, given that

$$A^* = \frac{\mathbb{E}(X \mathbf{1}_{W \le \xi})}{\mathbb{E}(X \mathbf{1}_{W > \xi})}, \tag{72}$$

the $\lim_{\alpha \to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} < 0$. Thus, there is $\delta > 0$ such that $\lim_{\alpha \to 0^+} \frac{d\mu(A,\alpha)]}{d\alpha} < 0$ for every $\alpha \in (0, \delta)$, $\mu(A, \alpha)$ is a strictly decreasing function with a local maximum at $\alpha = 0$ where $\mu(A, 0) = r$. Therefore,

$$\mu(A,\alpha) \le \mu(A,0), \forall\, \alpha \in (-\epsilon, \delta) \Rightarrow$$
$$U(\mu(A,\alpha)) \le U(\mu(A,0)) = U(r) \Rightarrow \max_{\alpha} U(\mu(A,\alpha)) = \max_{\alpha} U(\mu(A,0)) = U(r). \tag{73}$$

We should notice that if $A < A^*$, the weight α is positive. Indeed we obtain

$$\lim_{\alpha \to 0^+} \frac{d\mu(A,\alpha)}{d\alpha} > \frac{A^* \mathbb{E}(X \mathbf{1}_{W > \xi}) + A^* \mathbb{E}(X \mathbf{1}_{W \le \xi})}{A \mathbb{E}(X \mathbf{1}_{W > \xi}) + A^* \mathbb{E}(X \mathbf{1}_{W \le \xi})} = 0, \tag{74}$$

which implies that there exists

$$\delta > 0 : \frac{d\mu(A,\alpha)}{d\alpha} > 0, \forall \alpha \in (0,\delta) \Rightarrow \mu(A,\alpha) \uparrow (0,\delta). \tag{75}$$

Thus,

$$\begin{aligned} \mu(A,\alpha) \uparrow (-\epsilon,\delta) &\Rightarrow \exists\, \xi_0,\ \mu(A,\alpha) \uparrow [-\xi, \xi] \\ \max_{\alpha} \mu(A,\alpha) =&\mu(A,\ \xi),\ \xi > 0 \Rightarrow \alpha^* = \xi. \end{aligned} \tag{76}$$

■

D Bayesian Portfolio Analysis

This appendix includes the proofs to lemmas 1 and 2 regarding the posterior distriburion with i.i.d. and predictable returns respectively.

D.1 Proof of Lemma 1 (*i.i.d.* returns)

We present the results for the Bayesian portfolio with investment in a risk-free and a risky asset.[17] The investor models her excess returns based on the following equation

$$r_t = \mu + \epsilon_t, \tag{77}$$

where r_t is the continuously compounded excess return at time t and $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. disturbance terms.

In line with most of the relevant literature, to estimate the posterior distribution $p(\mu, \sigma^2|Y)$, we consider an uninformative prior of the form

$$p(\mu, \sigma)d\mu d\sigma \propto \frac{1}{\sigma} d\mu d\sigma. \tag{78}$$

The joint posterior of μ and σ is

$$p(\mu, \sigma|Y) \propto p(\mu, \sigma) \times L(\mu, \sigma|Y), \tag{79}$$

where $L(.)$ is the likelihood function. The joint posterior density for μ and σ follows a normal distribution and is also equal to

$$\begin{aligned}
p(\mu, \sigma|Y) &\propto \frac{1}{\sigma} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{(y_i - \mu)^2}{2\sigma^2}\right\} \\
&\propto \sigma^{-(n+1)} exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right\} \\
&= \sigma^{-(n+1)} exp\left\{-\frac{1}{2\sigma^2} n\left(\mu^2 + \frac{\sum_i y_i^2}{n} - \frac{2\mu \sum_i y_i}{n}\right)\right\}.
\end{aligned} \tag{80}$$

[17] The derivations for both cases (i.i.d. returns and predictable returns) follow closely the models of (Zellner, 1996; Tiao and Zellner, 1964) but they are reported in a more analytical way here as especially in the case of i.i.d. returns a number of steps is omitted in the original papers.

Completing the square, Eq. (80) can be written as

$$= \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2} n\left(\mu^2 - \frac{2\mu \sum_i y_i}{n} + \frac{\sum_i y_i^2}{n} + \left(\frac{2\sum_i y_i}{2n}\right)^2 - \left(\frac{2\sum_i y_i}{2n}\right)^2 \right) \right\}$$
$$= \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2} n\left(\left(\mu - \frac{\sum_i y_i}{n}\right)^2 + \frac{\sum_i y_i^2}{n} - 2\left(\frac{\sum_i y_i}{n}\right)^2 + \left(\frac{\sum_i y_i}{n}\right)^2 \right) \right\}$$
$$= \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2} \left(n\left(\mu - \frac{\sum_i y_i}{n}\right)^2 + \sum_i y_i^2 - 2\frac{\sum_i y_i}{n} \sum_i y_i + n\left(\frac{\sum_i y_i}{n}\right)^2 \right) \right\}. \tag{81}$$

Performing the substitution $\overline{\mu} = \frac{\sum_i y_i}{n}$, Eq. (81) can be rewritten as

$$\begin{aligned} p(\mu, \sigma|Y) &\propto \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2}\left(n(\mu - \overline{\mu})^2 + \sum_i y_i^2 - 2\overline{\mu}\sum_i y_i + n\overline{\mu}^2 \right) \right\} \\ &= \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2}(n(\mu - \overline{\mu})^2 + \sum_i (y_i - \overline{\mu}^2)) \right\}. \end{aligned} \tag{82}$$

Dividing $\sum_i (y_i - \overline{\mu}^2)$ by $n-1$ in Eq. (82) yields the unbiased variance estimator s^2 which gives the following:

$$p(\mu, \sigma|Y) \propto \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2}(n(\mu - \overline{\mu})^2 + (n-1)s^2) \right\}, \tag{83}$$

where $(n-1)s^2 = \sum_i (y_i - \overline{\mu})^2$. From Eq. (83), we see that the conditional mean and variance for the posterior mean are $E(\mu|\sigma, Y) = \overline{\mu}$ and $var(\mu|\sigma, Y) = \sigma^2/n$, respectively. To sample from the posterior for the mean, $p(\mu|Y, \sigma)$ conditional on σ and the sample data Y we use the normal $\mathcal{N}(\overline{\mu}, \sigma^2 + \sigma^2/n)$. As this expression is conditional on σ we calculate the marginal posterior distribution for the standard deviation and then using this result we draw from the normal for the mean. Following Zellner (1996), we marginalize out μ to derive the marginal posterior density for σ by expressing Eq. (82) as

$$\begin{aligned} p(\sigma|Y) &= \int_{-\infty}^{\infty} p(\mu, \sigma|Y) d\mu \\ &= \int_{-\infty}^{\infty} \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2}(n(\mu - \overline{\mu})^2 + (n-1)s^2) \right\} d\mu \\ &= \sigma^{-(n+1)} exp\left\{ -\frac{1}{2\sigma^2}(n-1)s^2 \right\} \int_{-\infty}^{\infty} exp\Big\{ -\underbrace{\frac{1}{2\sigma^2} n(\mu - \overline{\mu})^2}_{\phi^2/2} \Big\} d\mu. \end{aligned} \tag{84}$$

Setting the absolute value of the exponent inside the integral equal to $\phi^2/2$, we rewrite Eq. (84) as

$$p(\sigma|Y) \propto \sigma^{-(n+1)} exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \int_{-\infty}^{\infty} exp\left\{-\frac{\phi^2}{2}\right\} \sigma d\phi, \tag{85}$$

where the integration is now taking place w.r.t. ϕ as a result of the substitution, we performed. This follows from setting $\phi = \sigma^{-1}\sqrt{n}(\mu - \overline{\mu})$ which by differentiating both sides leads to $d\phi = \sigma^{-1}\sqrt{n}d\mu \propto \sigma^{-1}d\mu$ since everything else is constant. Next, we rewrite Eq. (85) as follows

$$\begin{aligned} p(\sigma|Y) &= \sigma^{-(n+1)}\sigma exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \int_{-\infty}^{\infty} exp\left\{-\frac{1}{2}\phi^2\right\} d\phi \\ &= \sqrt{2\pi}\sigma^{-n} exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \\ &\propto \sigma^{-n} exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\}. \end{aligned} \tag{86}$$

In the second equality we substitute the integral with its solution as it represents a Gaussian integral which has a known general solution given by $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$. Setting $n - 1 = N$ we have $n = N + 1$ and Eq. (86) can be written as

$$p(\sigma|Y) \propto \sigma^{-(N+1)} exp\left\{-\frac{Ns^2}{2\sigma^2}\right\}. \tag{87}$$

As Zellner (1996) observes, the PDF, Eq. (87), is proportional to the form of an inverse Gamma distribution with parameters $\alpha = N/2 = (n-1)/2$ and $\beta = Ns^2/2 = (N/2)(1/N)s^2 = 1/2s^2 = 1/2\sum_{i=1}^{n}(y_i - \overline{\mu}^2)$. The posterior distribution of the variance is now given by

$$\sigma^2|Y \sim Inv - Gamma\left(\frac{N}{2}, \frac{1}{2}\sum_{i=1}^{N+1}(y_i - \overline{\mu})^2\right). \tag{88}$$

Next, given the draw for σ we sample from the posterior of the mean

$$p(\mu|\sigma, Y) \sim \mathcal{N}\left(\overline{\mu}, \sigma^2/N\right). \tag{89}$$

We observe that Eq. (89) captures the dependence of the posterior mean on the size of the available dataset. As N becomes larger, the variance of μ becomes lower as a result of the smaller uncertainty around its true value which in turn stems from the more available data. To obtain an accurate approximation of the posterior distribution, we sample from Eqs. (88) and (89) a few hundred thousand times generating every time one value for μ and one value for σ. Now, for an investor who considers parameter uncertainty to sample from the predictive posterior we create a return value for each pair (μ, σ^2) (if we create $1,000,000$ pairs of μ and σ from Eqs. (88) and (89), we will generate $1,000,000$ return values, one for each pair). These returns $(R_1, \ldots, R_N)$ are the inputs to the Monte Carlo simulations we run to obtain the optimal weights.

The difference between an agent who considers parameter uncertainty and one who ignores it, lies in the way the returns are modelled; the latter creates new samples by drawing from a distribution with fixed parameter values while the former uses each time one of the pairs (μ, σ^2) generated by the sampling procedure.

D.2 Proof of Lemma 2 (Parameter Uncertainty with Predictable Returns)

We present the Bayesian framework for the case of returns predictable through the dividend yield. Under the assumption of normality and working with the compact form of the VAR as in Eq. (21), the likelihood of B, Σ given X, Z, where Σ is the residual positive-definite covariance matrix, takes the form of

$$\begin{aligned}
\mathcal{L}(B, \Sigma | X, Z) &= \frac{1}{\sqrt{((2\pi)^k |\Sigma|)^n}} exp\Big\{ -\frac{1}{2}\sum_{i=1}^{n}(X_i - B'Z_i)'\Sigma^{-1}(X_i - B'Z_i)\Big\} \\
&= \frac{1}{\sqrt{((2\pi)^k |\Sigma|)^n}} exp\Big\{ -\frac{1}{2} tr(X - BZ)(X - BZ)'\Sigma^{-1}\Big\} \\
&\propto |\Sigma|^{(-n/2)} exp\Big\{ -\frac{1}{2} tr(X - BZ)(X - BZ)'\Sigma^{-1}\Big\} \\
&= |\Sigma|^{(-n/2)} exp\Big\{ -\frac{1}{2} tr[S + (B - \hat{B})'Z'Z(B - \hat{B})]\Sigma^{-1}\Big\},
\end{aligned} \tag{90}$$

where tr is the the trace function. The last equality follows from $(X - BZ)'(X - BZ) = (X - \hat{B}Z)'(X - \hat{B}Z) + (B - \hat{B})'ZZ'((B - \hat{B})) = S + (B - \hat{B})'ZZ'((B - \hat{B}))$. A suitable uninformative prior given independence between B and Σ is the independence – Jeffreys prior given by

$$\begin{aligned} p(B,\Sigma) &= p(B)p(\Sigma) \\ &\propto |\Sigma|^{-(m+1)/2}, \end{aligned} \tag{91}$$

with $p(B)$ a constant. Now, combining the prior in Eq. (91) with the likelihood in Eq. (90) we derive the joint posterior for B and Σ

$$\begin{aligned} p(B,\Sigma|Z,X) &\propto |\Sigma|^{-(m+1)/2}|\Sigma|^{(-n/2)} exp\bigg\{-\frac{1}{2}tr[S+(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1}\bigg\} \\ &= |\Sigma|^{-(n+m+1)/2} exp\bigg\{-\frac{1}{2}tr[S+(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1}\bigg\}. \end{aligned} \tag{92}$$

We notice that Eq. (92) can be written in a similar form to the expression for the i.i.d. case as

$$p(B,\Sigma|Z,X) = p(B|\Sigma,Z,X) \times p(\Sigma|Z,X), \tag{93}$$

which is equal to

$$\begin{aligned} p(B,\Sigma|Z,X) \propto |\Sigma|^{-(n+m+1)/2} exp\bigg\{-\frac{1}{2}tr[(B-\hat{B})'Z'Z(B-\hat{B})]\Sigma^{-1}\bigg\} \\ \times\, exp\bigg\{\frac{1}{2}tr\, S\Sigma^{-1}\bigg\}. \end{aligned} \tag{94}$$

Chatfied and Collins (2013) show that we can split Eq. (94) as

$$p(B|\Sigma,Z,X) \propto |\Sigma|^{-k/2} exp\bigg\{-\frac{1}{2}[(\beta-\hat{\beta})'\Sigma^{-1}\otimes Z'Z(\beta-\hat{\beta})]\bigg\}, \tag{95}$$

where $\beta = (\beta_1,\ldots,\beta_m)$ is the matrix of regression coefficients ($\hat{\beta}$ are their estimates), $\otimes$ is the Kronecker product operator and $\nu = n-k+m+1$ and

$$p(\Sigma|Z,X) \propto |\Sigma|^{-\nu/2} exp\{tr\, S\Sigma^{-1}\}, \tag{96}$$

where $S = (X-\hat{B}Z)'(X-\hat{B}Z)$. It can be proved (Tiao and Zellner, 1964; Zellner, 1996) that the conditional posterior for B is in the form of a multivariate normal density function with mean $\hat{\beta}$ and covariance $\Sigma^{-1}\otimes Z'Z$ while the posterior predictive for Σ in Eq. (96) is distributed as

$$\Sigma \sim \mathcal{W}^{-1}((X-Z\hat{B})'((X-Z\hat{B}),T-n-1). \tag{97}$$

In Eq. (97), $\mathcal{W}^{-1}$ represents the inverse Wishart distribution with parameters the variance-covariance matrix scaled estimator and $T-n-1$ degrees of freedom. In order to facilitate and speed up the sampling procedure we inverse this relationship to obtain the distribution of the inverse variance covariance matrix. This follows now the distribution

$$\Sigma^{-1} \sim \mathcal{W}([(X-Z\hat{B})'((X-Z\hat{B})]^{-1}, T-n-1). \tag{98}$$

with the same degrees of freedom and the sampling parameter equal to the inverse of the variance covariance matrix estimator.

To sample from the posterior distribution we use a standardized procedure. We first sample for the variance – covariance matrix from $p(\Sigma^{-1}|X)$ and then given this draw we sample for the AR matrix and the constant coefficients from the $p(vec(B)|\Sigma^{-1}, X) = \mathcal{N}(vec(\hat{B}), \Sigma^{-1} \otimes Z'Z)$. Given these sets of parameters we simulate forward the VAR to obtain a large number ($\geq 10,000$) of future stock return paths. This specification captures the uncertainty in stock returns' forecasts since the VAR parameters are not taken as the true ones as in the case where parameter uncertainty is ignored. An investor who uses the latter approach simulates forward the VAR based on fixed parameters obtained by the calibration using observed data.

In the next step of the sampling procedure we calculate the mean and variance of the first two moments of the return paths generated in step one. Based on these statistics we sample for the return values and their variance which are now normally distributed, with each draw representing a quarterly return and variance. In our case, given the 40–year horizon, we sample $40 \times 4 = 160$ points, which are the inputs to the dynamic programming algorithm to solve the DA portfolio problem.

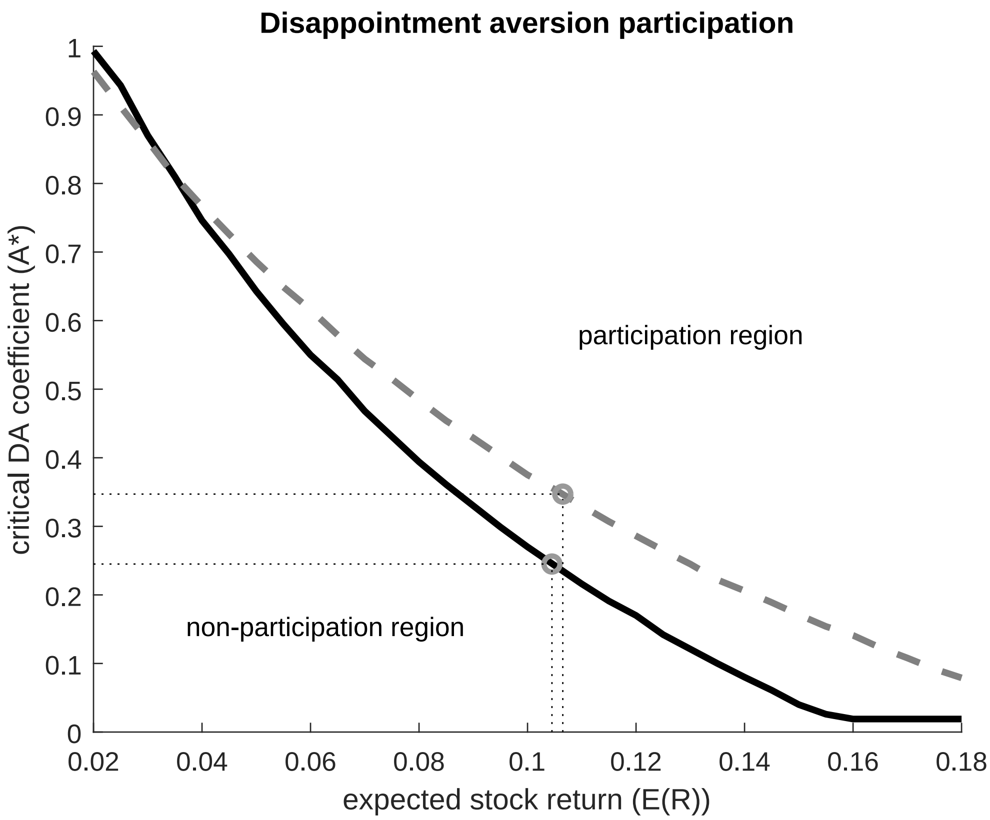

Figure 1: Stock market participation/non-participation regions with DA preferences. The graph shows how the expected level of stock returns (stated annually) affects the critical level of the DA coefficient (A^{*}). Two lines are presented: the solid one corresponds to the critical DA coefficients for the dataset used in our study (1934-2016) and the dashed line plots the critical DA values for the dataset used in Ang et al. (2005). The grey squares represent the critical DA level (A^{*} which induces non-participation) which correspond to the historical mean of the equity return for the two data samples.

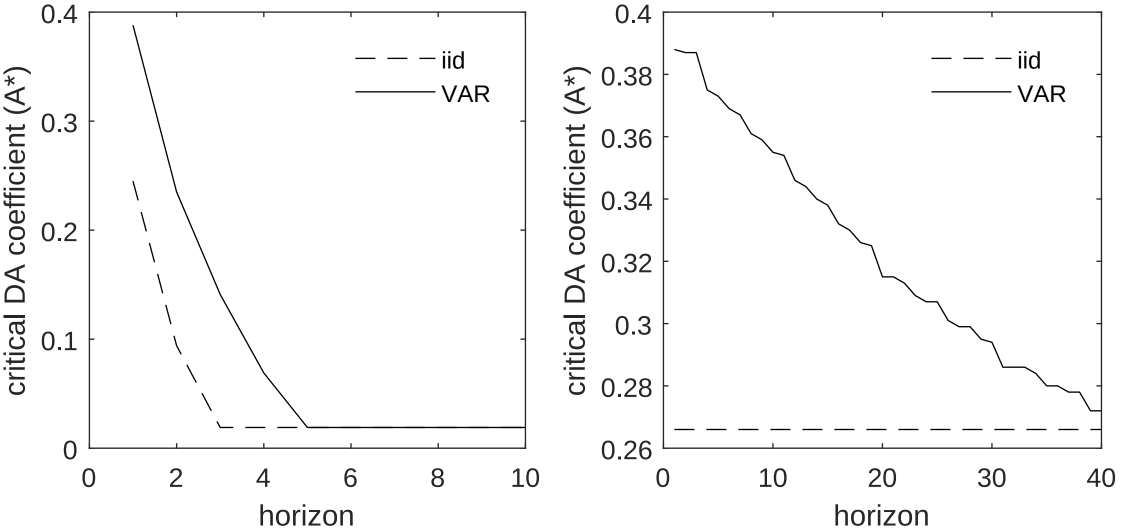

Figure 2: Critical DA level (A^*) that induces non–participation in the stock market for a buy–and–hold investor (left graph) and a dynamic investor (right graph). The dashed line corresponds to the case of i.i.d. returns (normality and non–predictability) while the solid line corresponds to the case of predictable returns. Investors would invest in the stock market when their DA coefficient lies in the area above the lines. To display the graphs more clearly, the one on the left (buy–and–hold) plots the A^* for a period up to 10 years as beyond that point A^* remains constant and very close to zero.

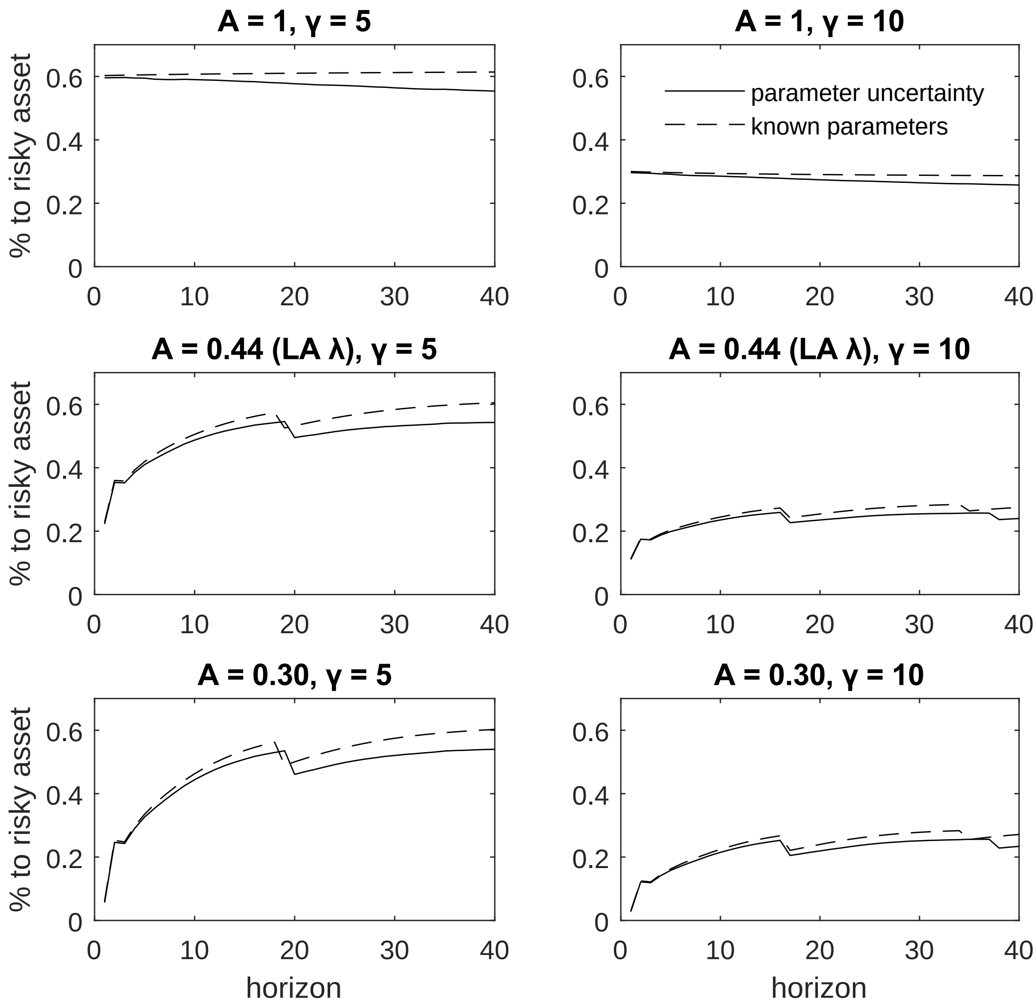

Figure 3: Optimal portfolio allocation to the risky asset for an investor who follows a buy-and-hold investment strategy, uses the i.i.d. return generator and either incorporates (solid line) or ignores (dashed line) uncertainty in model parameters. The investor in the top row uses a CRRA (i.e. power) utility function with two levels of risk aversion while the other two cases (middle and bottom row) make use of the DA utility function with two different values for the DA coefficient. $A = 0.44$ is equivalent to the value of the Loss Aversion (LA) parameter calculated in Tversky and Kahneman (1992), i.e., $DA = 1/\lambda = 0.44$. We observe that a DA investor holds a significantly different portfolio to one who uses a power utility function.

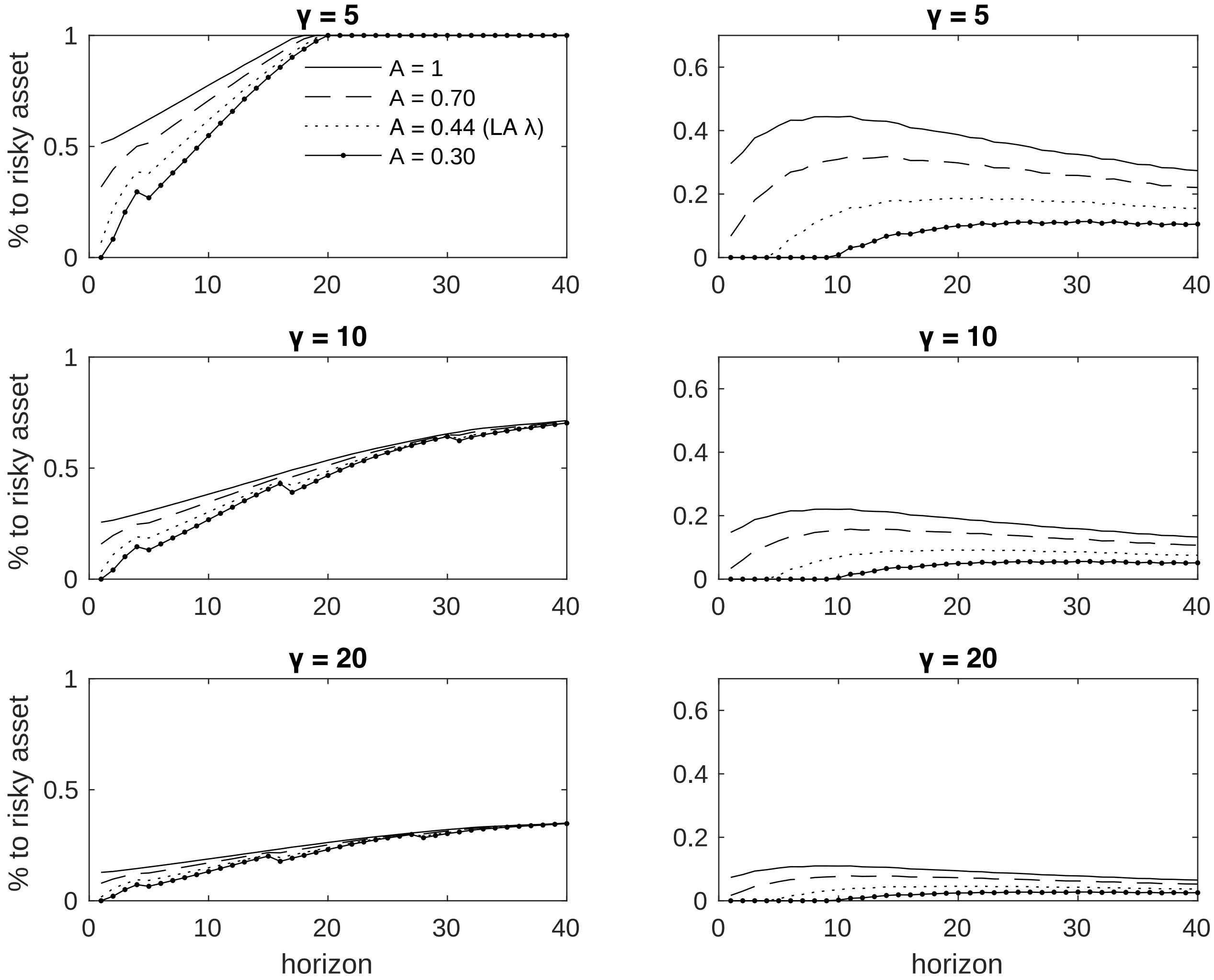

Figure 4: Optimal portfolio allocation to equities for different horizons when the VAR is used to forecast equity returns. The investor follows a buy–and–hold strategy by choosing the portfolio allocation to the risky asset in the beginning of the investment period. $A = 0.44$ is equivalent to the value of the Loss Aversion (LA) parameter calculated in Tversky and Kahneman (1992), i.e., $DA = 1/\lambda = 0.44$. The graphs on the left column ignore parameter uncertainty while the ones on the right account for this. Three levels of risk aversion and four levels of disappointment aversion are represented.

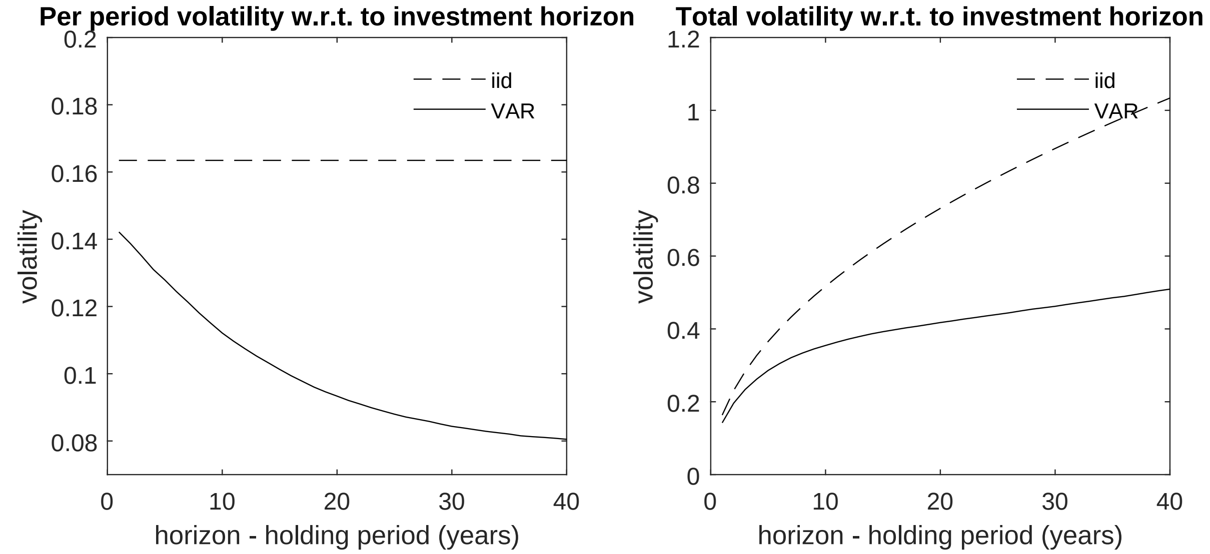

Figure 5: Evolution of per-period and long-term volatility for the risky asset. The dotted line corresponds to the case of an investor who models returns as iid while the solid line shows the volatility for an investor who uses the VAR to forecast equity returns.

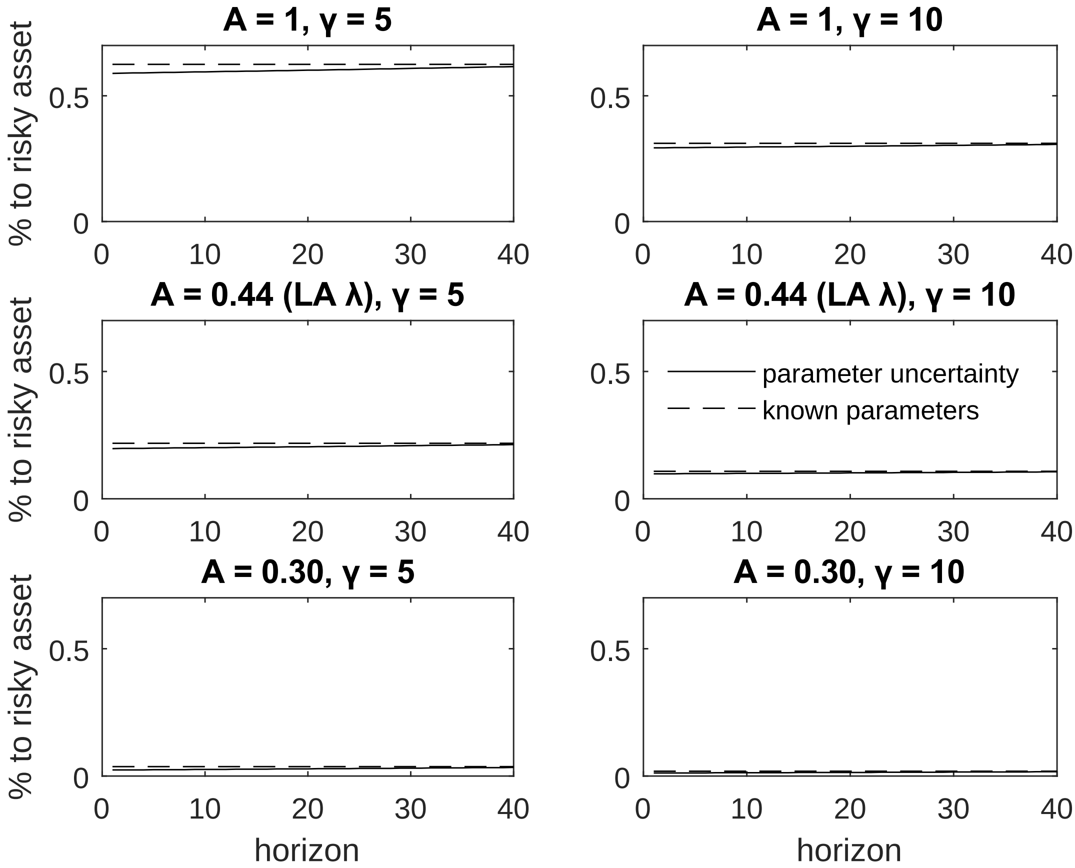

Figure 6: Dynamic portfolio allocation between the risky and the riskless asset for an investor who uses the i.i.d. return generator for the risky asset. The objective of this exercise is to show how the portfolio allocation to the risky asset changes for an investor who acknowledges parameter uncertainty (solid line) compared to one who ignores it (dashed line) and holds the same portfolio throughout the investment horizon. $A = 0.44$ is equivalent to the value of the Loss Aversion (LA) parameter calculated in Tversky and Kahneman (1992), i.e., $DA = 1/\lambda = 0.44$.

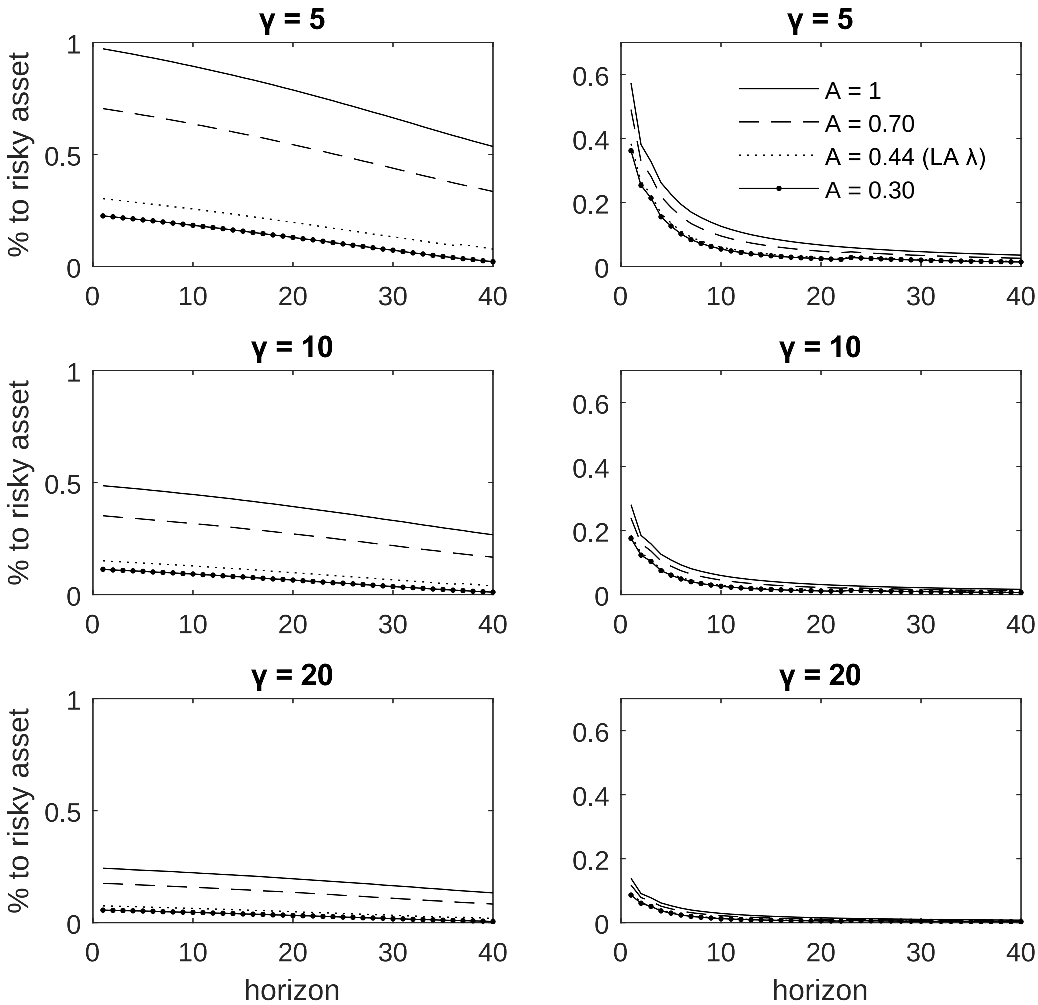

Figure 7: Optimal portfolio allocation at different time horizons for an investor who follows a dynamic reallocation using the VAR to forecast returns. The left columns reports results when parameter uncertainty is ignored while the one on the right takes parameter uncertainty into account. Each line corresponds to a different level of the DA coefficient (A) as follows: solid line, A = 1; dashed line, A = 0.70; dotted line, A = 0.44; solid/dotted line, A = 0.30. $A = 0.44$ is equivalent to the value of the Loss Aversion (LA) parameter calculated in Tversky and Kahneman (1992), i.e., $DA = 1/\lambda = 0.44$.

Table 1
Summary statistics

	S&P 500	3-month T-bill	Excess Return
		Annualized	
mean	0.1045	0.0344	0.0695
stdev	0.1625	0.0088	0.1644
		Quarterly	
mean	0.0251	0.0085	0.0166
stdev	0.0817	0.0044	0.0822

S&P 500 and T-bill summary statistics annualized. Excess return is calculated by subtracting the 3-month T-bill rate from the value of the S &P 500 for the same period.

Table 2
Parameter estimates for the Data Generating Process (VAR)

Parameter	With predictability	Without predictability
c_1	0.1222	0.0128
	(0.0173)	(0.0178)
c_2	-0.0004	-0.0317
	(0.0119)	(0.0150)
b_{11}	0.0259	0.0
	0.1176	–
b_{12}	0.0220	0.0
	(0.1354)	–
b_{21}	-0.7068	0.0
	(0.0807)	–
b_{22}	0.9978	0.9932
	(0.0929)	(0.0912)
σ_{11}	0.0850	0.0856
	(0.0037)	(0.0042)
σ_{22}	0.0408	0.0752
	(0.0017)	(0.0029)
ρ	-0.5216	-0.2980
	(0.0021)	(0.0028)

VAR estimation and corresponding standard errors of the parameters for the two systems (predictability/no-predictability). The model in Eq. (20) is estimated with the method of Maximum likelihood (MLE). In the case of the non-predictability system the autoregressive coefficient matrix is set to zero, while when we account for predictability in returns, all four coefficients are free to vary without restrictions. Parentheses include the standard errors of the estimated coefficients. For the S&P500 index and the dividend yield quarterly data from the period January 1934 to September 2016 are used in our calculations.